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GRE MATH PROBLEMS

arranged by **Topic** and **Difficulty** Level

By Dr. Steve Warner

320 Level 1, 2, 3, 4, and 5 Math Problems for the GRE

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320 GRE Math Problems arranged by Topic and Difficulty Level

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Dr. Steve Warner

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INTRODUCTION
THE PROPER WAY TO PREPARE

There are many ways that a student can prepare for the GRE. But not all preparation is created equal. I always teach my students the methods that will give them the maximum result with the minimum amount of effort.

The book you are now reading is self-contained. Each problem was carefully created to ensure that you are making the most effective use of your time while preparing for the GRE. By grouping the problems given here by level and topic I have ensured that you can focus on the types of problems that will be most effective to improving your score.

1. Using this book effectively

- Begin studying at least three months before the GRE
- Practice GRE math problems twenty minutes each day
- Choose a consistent study time and location

You will retain much more of what you study if you study in short bursts rather than if you try to tackle everything at once. So try to choose about a twenty-minute block of time that you will dedicate to GRE math each day. Make it a habit. The results are well worth this small time commitment.

- Every time you get a question wrong, **mark it off, no matter what your mistake**.
- Begin each study session by first redoing problems from previous study sessions that you have marked off.
- If you get a problem wrong again, **keep it marked off**.

Note that this book often emphasizes solving each problem in more than one way. Please listen to this advice. The same question is not generally repeated on any GRE so the important thing is learning as many techniques as possible.

Being able to solve any specific problem is of minimal importance. The more ways you have to solve a single problem the more prepared you will be to tackle a problem you have never seen before, and the quicker you will be able to solve that problem. Also, if you have multiple methods for solving a single problem, then on the actual GRE when you "check over" your work you will be able to redo each problem in a different way. This will eliminate all "careless" errors on the actual exam. Note that in this book the quickest solution to any problem will always be marked with an asterisk (*).

2. The magical mixture for success

A combination of three components will maximize your GRE math score with the least amount of effort.

- Learning test taking strategies that work specifically for standardized tests.
- Practicing GRE problems for a small amount of time each day for about three months before the GRE.
- Taking about four practice tests before test day to make sure you are applying the strategies effectively under timed conditions.

I will discuss each of these three components in a bit more detail.

Strategy: The more GRE specific strategies that you know the better off you will be. Throughout this book you will see many strategies being used. Some examples of basic strategies are "plugging in answer choices," "taking guesses," and "picking numbers." Some more advanced strategies include "trying a simple operation," and "moving the sides of a figure around." Pay careful attention to as many strategies as possible and try to internalize them. Even if you do not need to use a strategy for that specific problem, you will certainly find it useful for other problems in the future.

Practice: The problems given in this book are more than enough to vastly improve your current GRE math score. All you need to do is work on these problems for about ten to twenty minutes each day over a period of three to four months and the final result will far exceed your expectations.

Let me further break this component into two subcomponents – **topic** and **level**.

Topic: You want to practice each of the four general math topics given on the GRE and improve in each independently. The four topics are **Arithmetic**, **Algebra**, **Geometry**, and **Data Analysis**. The problem sets in this book are broken into these four topics.

Level: You will make the best use of your time by primarily practicing problems that are at and slightly above your current ability level. For example, if you are struggling with Level 2 Geometry problems, then it makes no sense at all to practice Level 5 Geometry problems. Keep working on Level 2 until you are comfortable, and then slowly move up to Level 3. Maybe you should never attempt those Level 5 problems. You can get an exceptional score without them (higher than a 700).

Tests: You want to take about four practice tests before test day to make sure that you are implementing strategies correctly and using your time wisely under pressure. For this task you should use actual GRE exams such as those found in "The Official Guide to the GRE Revised General Test." Take one test every few weeks to make sure that you are implementing all the strategies you have learned correctly under timed conditions. If you will be taking the computer-based test, make sure you do at least two practice tests on the computer. You can download practice tests for free here:

http://www.ets.org/gre/revised_general/prepare/powerprep2

3. Practice problems of the appropriate level

Roughly speaking about one third of the math problems on the GRE are easy, one third are medium, and one third are hard. If you answer two thirds of the math questions on the GRE correctly, then your score will be approximately a 160 (out of 170). That's right—you can get about a 160 on the math portion of the GRE without answering a single hard question.

Keep track of your current ability level so that you know the types of problems you should focus on. If you are currently scoring around a 145 on your practice tests, then you should be focusing primarily on Level 1, 2, and 3 problems. You can easily raise your score 6 to 8 points without having to practice a single hard problem.

If you are currently scoring about a 155, then your primary focus should be Level 2 and 3, but you should also do some Level 1 and 4 problems.

If you are scoring around a 160, you should be focusing on Level 2, 3, and 4 problems, but you should do some Level 1 and 5 problems as well.

Those of you at the 165 level really need to focus on those Level 4 and 5 problems.

If you really want to refine your studying, then you should keep track of your ability level in each of the four major categories of problems:

- **Arithmetic**
- **Algebra**
- **Geometry**
- **Data Analysis**

For example, many students have trouble with very easy Geometry problems, even though they can do more difficult Arithmetic problems. This type of student may want to focus on Level 1, 2, and 3 Geometry questions, but Level 3 and 4 Arithmetic questions.

4. Practice in small amounts over a long period of time

Ideally you want to practice doing GRE math problems twenty minutes each day beginning at least 3 months before the exam. You will retain much more of what you study if you study in short bursts than if you try to tackle everything at once.

The only exception is on a day you do a practice test. You should do at least four practice tests before you take the GRE. At first you can do just the math sections. The last one or two times you take a practice test you should do the whole test in one sitting. As tedious as this is, it will prepare you for the amount of endurance that it will take to get through this exam.

So try to choose about a twenty-minute block of time that you will dedicate to GRE math every night. Make it a habit. The results are well worth this small time commitment.

5. Redo the problems you get wrong over and over and over until you get them right

If you get a problem wrong, and never attempt the problem again, then it is extremely unlikely that you will get a similar problem correct if it appears on the GRE.

Most students will read an explanation of the solution, or have someone explain it to them, and then never look at the problem again. This is *not* how you optimize your GRE score. To be sure that you will get a similar problem correct on the GRE, you must get the problem correct before the GRE—and without actually remembering the problem.

This means that after getting a problem incorrect, you should go over and understand why you got it wrong, wait at least a few days, then attempt the same problem again. If you get it right, you can cross it off your list of problems to review. If you get it wrong, keep revisiting it every few days until you get it right. Your score *does not* improve by getting problems correct. **Your score improves when you learn from your mistakes.**

6. Check your answers properly

When you go back to check your earlier answers for careless errors *do not* simply look over your work to try to catch a mistake. This is usually a waste of time. Always redo the problem without looking at any of your previous work. Ideally, you want to use a different method than you used the first time.

For example, if you solved the problem by picking numbers the first time, try to solve it algebraically the second time, or at the very least pick different numbers. If you do not know, or are not comfortable with a different method, then use the same method, but do the problem from the beginning and do not look at your original solution. If your two answers do not match up, then you know that this a problem you need to spend a little more time on to figure out where your error is.

This may seem time consuming, but that's okay. It is better to spend more time checking over a few problems than to rush through a lot of problems and repeat the same mistakes.

7. Take a guess whenever you cannot solve a problem

There is no guessing penalty on the GRE. Whenever you do not know how to solve a problem take a guess. Ideally you should eliminate as many answer choices as possible before taking your guess, but if you have no idea whatsoever do not waste time overthinking. Simply put down an answer and move on. You should certainly mark it off and come back to it later if you have time.

8. Pace yourself

Do not waste your time on a question that is too hard or will take too long. After you've been working on a question for about 30 to 45 seconds you need to make a decision. If you understand the question and think that you can get the answer in another 30 seconds or so, continue to work on the problem. If you still do not know how to do the problem or you are using a technique that is going to take a long time, mark it off and come back to it later if you have time.

If you do not know the correct answer, eliminate as many answer choices as you can and take a guess. But you still want to leave open the possibility of coming back to it later. Remember that every problem is worth the same amount. Do not sacrifice problems that you may be able to do by getting hung up on a problem that is too hard for you.

9. Attempt the right number of questions

Many students make the mistake of thinking that they have to attempt every single GRE math question when they are taking the test. There is no such rule. In fact, most students will increase their GRE score by *reducing* the number of questions they attempt. The following chart gives a general guideline for how many questions you should be attempting in each math section. The leftmost column is your latest score on an official practice test. The middle column shows how many questions you should answer in each math section if you are taking the computer based test. The rightmost column shows how many questions you should answer in each math section if you are taking the paper based test.

Score	Computer	Paper
130 – 140	7/20	8/25
141 – 144	10/20	12/25
145 – 148	12/20	15/25
149 – 152	14/20	17/25
153 – 156	16/20	20/25
157 – 161	18/20	23/25
162 – 170	20/20	25/25

For example, a student with a current score of 155 on the computer based test should attempt about 16 questions in each math section. This is *just* a general guideline. Of course it can be fine-tuned. As a simple example, if you are particularly strong at Arithmetic problems, but very weak at Geometry problems, then you may want to try every Arithmetic problem no matter where it appears, and you may want to reduce the number of Geometry problems you attempt.

Remember that there is no guessing penalty on the GRE, so you should *not* leave any questions blank. This *does not* mean you should attempt every question. It means that if you are running out of time make sure you fill in answers for all the questions you did not have time to attempt.

For example, if you are currently scoring a 151 on the computer based test, then it is possible you will only be attempting about 14 questions in each section. Therefore, when you are running out of time you should fill in answers for the remaining 6 problems. If you happen to get a chance to attempt some of them, you can always change your answer. But make sure those answers are filled in before the test ends!

10. Use the calculator wisely.

- The calculator provided to you should be used for basic arithmetic computations only: addition, subtraction, multiplication, division, and taking square roots.
- For most questions you should not need to use the calculator.
- I recommend performing just one operation at a time with the calculator, rather than using parentheses and trying to perform several computations at once.

11. Be familiar with the four question types.

There are four types of math questions given on the GRE:

- Comparison questions
- Multiple choice questions – select one answer choice
- Multiple choice questions – select one or more answer choices
- Numeric entry questions (also called free response questions or grid in questions)

Comparison questions ask you to compare two quantities. The answer choices are always the same and they should be memorized before taking the exam.

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Multiple choice questions ask you to select one or more answers from a list of choices. Some of these questions will indicate how many choices you should select, and others will not.

Numeric entry questions do not provide choices. You answer these questions by filling in circles in a grid. There are two types of grids: there is a grid that expects an integer or decimal answer, and there is a grid that expects a fraction as an answer.

Throughout this book you will learn many strategies to help you answer all of these different question types.

Problems by Level and Topic with Fully Explained Solutions

Note: The quickest solution will always be marked with an asterisk (*).

Level 1: Arithmetic

1. Quantity A: 5^4
   Quantity B: 6^3

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

*** Calculator solution:** $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$ and $6^3 = 6 \cdot 6 \cdot 6 = 216$.

So 5^4 is greater than 6^3, choice A.

Note: The expression 5^4 means to multiply the number 5 by itself 4 times. We can do this with the calculator.

Similarly, 6^3 means to multiply the number 6 by itself 3 times.

2. Quantity A: The number of days in 9 weeks
   Quantity B: The number of months in 5 years

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

* Since there are 7 days in a week, in 9 weeks there are $7 \cdot 9 = 63$ days.

Since there are 12 months in a year, in 5 years there are $12 \cdot 5 = 60$ months.

So the number of days in 9 weeks is greater than the number of months in 5 years, choice A.

In a decimal number, a bar over one or more consecutive digits means that the pattern of digits under the bar repeats without end. For example, $0.\overline{123} = 0.123123123\ldots$

3. Quantity A: $0.\overline{43}$

   Quantity B: $0.\overline{434}$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

*

$$0.\overline{43} = 0.434\mathbf{3}43\ldots$$
$$0.\overline{434} = 0.434\mathbf{4}34434\ldots$$

We can compare two decimals by looking at the first position where they disagree. Notice that the first place these two numbers disagree is the fourth position after the decimal point (marked in bold above). In $0.\overline{43}$ this digit is a 3 and in $0.\overline{434}$ this digit is a 4. So $0.\overline{434}$ is greater, choice B.

$s = 3.02973$ and s^* is the decimal expression for s rounded to the nearest hundredth.

4. Quantity A: The number of places where s and s^* differ

   Quantity B: 3

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

* $s^* = 3.03 = 3.03000$. So we see that s and s^* differ in 4 places. So Quantity A is greater, choice A.

Notes: (1) In the number 3.02973, the leftmost 3 is in the **ones** place, the 0 is in the **tenths** place, the 2 is in the **hundredths** place, the 9 is in the **thousandths** place, the 7 is in the **ten thousandths** place, and the rightmost 3 is in the **hundred thousandths** place.

(2) Since we are being asked to round s to the nearest hundredth, we focus on the 2 which is in the hundredths place. We look at the next position to the right (the thousandths place) to determine if we round up or down. Since that digit is at least 5 (more specifically it is 9) we round the 2 up to a 3.

(3) s and s^* differ in the hundredths, thousandths, ten thousandths, and hundred thousandths places.

(4) The **tenths** place is different from the **tens** place. For example, in the number 235.46, the digit 3 is in the tens place, whereas the digit 4 is in the tenths place. Similarly, the digit 2 is in the hundreds place, whereas the digit 6 is in the hundredths place.

5. $(34 - 33 - 32 - 31 - 30) - (35 - 34 - 33 - 32 - 31) =$

   A. -60
   B. -10
   C. -6
   D. 2
   E. 3

*** Quick solution:** -33, -32, and -31 appear in both pairs of parentheses. Since we are subtracting we can delete those numbers to get $(34 - 30) - (35 - 34) = 4 - 1 = 3$, choice E.

Solution using the distributive property: We eliminate the parentheses by distributing the subtraction symbol to get

$$34 - 33 - 32 - 31 - 30 - 35 + 34 + 33 + 32 + 31.$$

We can now cancel the 33's, 32's, and 31's to get

$34 - 30 - 35 + 34 = 4 - 35 + 34 = -31 + 34 = 3$, choice E.

Note: We can also solve this by direct computation with the help of the calculator. Be careful when using the calculator for this one, as it is very easy to make an error.

6. Of the following, which is closest to $\sqrt[3]{70}$

   A. 2
   B. 3
   C. 4
   D. 5
   E. 6

*** Solution by starting with choice C:** We start with choice C and compute $4^3 = 4 \cdot 4 \cdot 4 = 64$. This is a little less than 70.

Let's try D to be safe: $5^3 = 5 \cdot 5 \cdot 5 = 125$. This is much further from 70 than 64. So the answer is choice C.

Notes: (1) $70 - 64 = 6$, so that 64 is 6 units away from 70.

Also, $125 - 70 = 55$, so that 125 is 55 units away from 70.

Since 64 is much closer to 70, it follows that $4 = \sqrt[3]{64}$ is closer to $\sqrt[3]{70}$.

(2) When plugging in answer choices, it's always a good idea to start with choice C unless there is a specific reason not to. In this problem, by trying choice C, we were able to eliminate choices A and B right away, possibly saving some time.

7. Which of the following numbers is less than 0.216? Indicate <u>all</u> such values.

A. 0.2106
B. 0.2159
C. 0.2161
D. 0.2166
E. 0.22
F. 0.221

* We can compare two decimals by looking at the first position where they disagree. For example, 0.215 is less than 0.216 because 5 is less than 6. If a digit is missing, there is a hidden 0 there. Thus, 0.2 is also less than 0.216 because 0.2 is the same as 0.200 and 0 is less than 1 (remember that we look at the **first** position where the decimals disagree).

Using the above reasoning we see that 0.2159 is less than 0.216 and 0.2161 is greater than 0.216. Since the answers are listed in increasing order, the answers are choices A and B.

8. Each of A, B, C, D and E are distinct numbers from the set $\{2, 15, 25, 31, 34\}$ such that A is prime, B is even, C and D are multiples of 5, and $A < E < B$. What is E ?

* Remember that a prime number is a positive integer with **exactly** 2 factors (1 and itself). Since A is prime, it is either 2 or 31. Since B is even and greater than A it must be 34. Since C and D are multiples of 5 they must be 15 and 25 (not necessarily in that order). So E must be $\mathbf{31}$.

Definitions: The **integers** are the counting numbers together with their negatives.

$$\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

The **positive integers** are the positive numbers from this set.

$$\{1, 2, 3, 4, 5, 6, \dots\}$$

A **prime number** is a positive integer that has **exactly** two factors (1 and itself). Here is a list of the first few primes:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \dots$$

Note that 1 is **not** prime. It has only one factor!

A **composite number** has **more** than two factors. Here is a list of the first few composites:

$$4, 6, 8, 9, 10, 12, 14, 15, 16, \dots$$

LEVEL 1: ALGEBRA

$$z = -6$$

9. Quantity A: $5z^2$
   Quantity B: 180

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

* $5z^2 = 5(-6)^2 = 5(-6)(-6) = 5 \cdot 36 = 180$. So the two quantities are equal, choice C.

$$b < -2$$

10. Quantity A: $b + 1$
    Quantity B: -2

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

Algebraic solution: We add 1 to each side of the given inequality to get $b + 1 < -2 + 1 = -1$.

So, for example, $b + 1$ could be -2 in which case Quantities A and B would be equal.

On the other hand, $b + 1$ could be -3 in which case Quantities A and B would not be equal.

So the answer is choice D.

Computation in detail: We add one to each side of the given inequality:

$$\begin{array}{r} b < -2 \\ \underline{+1 \quad +1} \\ b + 1 < -1 \end{array}$$

*** Solution by picking numbers:** If $b = -3$, then $b + 1 = -2$, and Quantities A and B are equal.

If $b = -4$, then $b + 1 = -3$, and Quantities A and B are not equal.

So the answer is choice D.

$$y = 3x - 5$$
$$x = -2$$

11. Quantity A: y
    Quantity B: -11

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Algebraic solution:** Since $x = -2$, we have

$$y = 3x - 5 = 3(-2) - 5 = -6 - 5 = -11$$

So Quantities A and B are equal, choice C.

x is an integer greater than 0

12. Quantity A: $3x + 7$
    Quantity B: $7x + 3$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If $x = 1$, then $3x + 7 = 3 \cdot 1 + 7 = 10$ and $7x + 3 = 7 \cdot 1 + 3 = 10$. So Quantities A and B are equal.

If $x = 2$, then we have $3x + 7 = 3 \cdot 2 + 7 = 6 + 7 = 13$ and also $7x + 3 = 7 \cdot 2 + 3 = 14 + 3 = 17$. So Quantities A and B are not equal.

So the answer is choice D.

13. If $2t = 8$ and $3s + t = 13$, what is the value of s?

    A. 2
    B. 3
    C. 4
    D. 5
    E. 6

Solution by starting with choice C: Looking at the first equation we see that t must be 4 (since $2 \cdot 4 = 8$). Substituting $t = 4$ into the second equation we get

$$3s + 4 = 13$$

Now let's start with choice C as our first guess. We substitute 4 in for s.

$$3 \cdot 4 + 4 = 13$$
$$12 + 4 = 13$$
$$16 = 13$$

Since 16 is too big we can eliminate choices C, D and E.

We next try choice B and substitute 3 in for s.

$$3 \cdot 3 + 4 = 13$$
$$9 + 4 = 13$$
$$13 = 13$$

Since we get a true statement, the answer is choice B.

*** Algebraic solution:** Solving the first equation for t we get $t = 4$ (because $t = \frac{8}{2} = 4$). Substituting $t = 4$ into the second equation we get

$$3s + 4 = 13$$
$$3s = 9$$
$$s = 3$$

Thus, the answer is choice B.

Remark: The more advanced student should be able to do all of these computations in his/her head.

14. If $3 + x + x + x = 1 + x + x + x + x + x$, what is the value of x?

A. 1
B. 2
C. 3
D. 4
E. 5

Solution by starting with choice C: Begin by looking at choice C. We substitute 3 in for x on both sides of the equation.

$$3 + 3 + 3 + 3 = 1 + 3 + 3 + 3 + 3 + 3$$
$$12 = 16$$

Since this is false, we can eliminate choice C. A little thought allows us to eliminate choices D and E as well. We'll try choice B next.

$$3 + 2 + 2 + 2 = 1 + 2 + 2 + 2 + 2 + 2$$
$$9 = 11$$

Finally, let us check that choice A is correct.

$$3 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1$$
$$6 = 6$$

Thus, the answer is choice A.

Algebraic solution:

$$3 + x + x + x = 1 + x + x + x + x + x$$
$$3 + 3x = 1 + 5x$$
$$2 = 2x$$
$$1 = x$$

Thus, the answer is choice A.

Remark: We can begin with an algebraic solution, and then switch to the easier method. For example, we can write $3 + 3x = 1 + 5x$, and then start substituting in the answer choices from here. This will take less time than the first method, but more time than the second method.

*** Striking off x's:** When the same term appears on both sides of an equation we can simply delete that term from both sides. In this problem we can strike off 3 x's from each side to get $3 = 1 + x + x$. This becomes $2 = 2x$ from which we see that $x = 1$, choice A.

15. John has fewer nickels than Phil, but more than Thomas. If J, P and T represent the number of nickels that each boy has, respectively, which of the following can be true? Indicate <u>all</u> such inequalities.

A. $J < P < T$
B. $J < T < P$
C. $P < J < T$
D. $P < T < J$
E. $T < J < P$
F. $T < P < J$

* When using the symbols "$<$" and "$>$", the symbol always points to the smaller number. We will use only the symbol "$<$" since this is the only symbol that appears in the answer choices. Since John has fewer nickels than Phil we have $J < P$. Since John has more nickels than Thomas we have $T < J$. Putting these two together gives us $T < J < P$. Thus, the only answer is choice E.

Remark: It might seem more natural to write $J > T$ because of the wording in the problem. This is fine, but you then just need to realize that $T < J$ means the same thing. Note again that in the end we want to have only the symbol "$<$" because this is the only symbol appearing in the answer choices.

16. If $5x - 4 = 26$, then $7x =$

*** Algebraic solution:** We add 4 to each side of the given equation to get $5x = 26 + 4 = 30$. Dividing each side of this last equation by 5 gives us $x = \frac{30}{5} = 6$. So $7x = 7 \cdot 6 = \mathbf{42}$.

Notes: (1) Be careful to finish the problem here. The question is asking for $7x$, and not just x.

(2) Instead of solving the first equation algebraically, we can also find x by taking guesses.

For example, if we take a guess that $x = 3$, then we have

$$5x - 4 = 5 \cdot 3 - 4 = 15 - 4 = 11.$$

This is too small. So we need to guess a larger number.

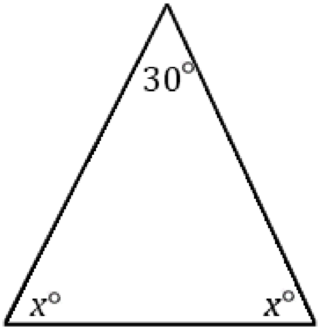
If we then try $x = 6$, we have

$$5x - 4 = 5 \cdot 6 - 4 = 30 - 4 = 26.$$

This is correct, and so $x = 6$.

It follows that $7x = 7 \cdot 6 = \mathbf{42}$.

LEVEL 1: GEOMETRY



17. Quantity A: x
    Quantity B: 75

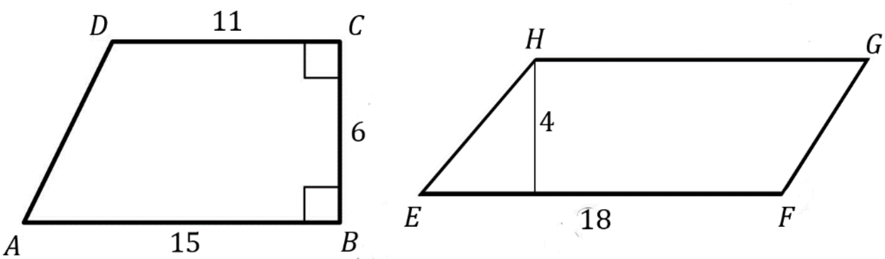
    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

Solution by trying Quantity B: Let's try Quantity B for x. That is we will set x equal to 75. It follows that the sum of the angle measures of the triangle is $x + x + 30 = 75 + 75 + 30 = 180$. This is exactly what the angle measures of a triangle are supposed to add up to. So x is in fact 75 and the two Quantities are equal, choice C.

* **Algebraic solution:** Using the fact that the angle measures of a triangle sum to 180° we have

$$x + x + 30 = 180$$
$$2x + 30 = 180$$
$$2x = 150$$
$$x = 75$$

So the two Quantities are equal, choice C.



18. Quantity A: The area of trapezoid *ABCD*
Quantity B: The area of parallelogram *EFGH*

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* The area of the trapezoid is $\frac{1}{2}(11 + 15)(6) = 78$ and the area of the parallelogram is $18 \cdot 4 = 72$. So Quantity A is greater than Quantity B, choice A.

Notes: (1) The area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$ where b_1 and b_2 are the two bases of the trapezoid and h is the height of the trapezoid.

In other words, to find the area of a trapezoid, we first find the average of the two bases, and then multiply this result by the height.

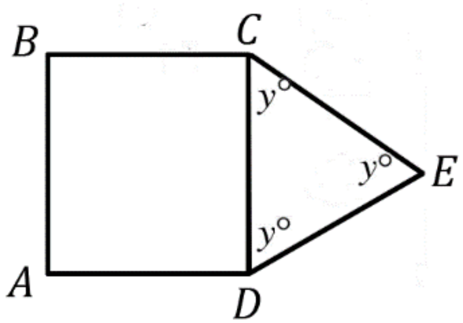
In trapezoid $ABCD$ the bases have lengths 11 and 15, and the height has length 6.

(2) The area of a parallelogram is $A = bh$ where b is the base of the parallelogram and h is the height of the parallelogram.

In parallelogram $EFGH$ the base has length 18, and the height length 4.

(3) For both figures, the height is always the perpendicular distance between bases.

Interesting observation: The formulas for the area of a trapezoid and the area of a parallelogram are actually the same. The formula for the parallelogram looks simpler because both bases have the same length. So when we take their average, we just get the common number back (for example, the average of 18 and 18 is $\frac{18+18}{2} = \frac{36}{2} = 18$).



Square $ABCD$ has area 25

19. Quantity A: The perimeter of pentagon $ADECB$
Quantity B: 30

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* Since the square has area 25, the side length of the square is $\sqrt{25} = 5$. Since all angles of the triangle are equal, the triangle is equilateral, and so each side of the triangle also has length 5. It follows that the perimeter of $ADECB$ is $5 \cdot 5 = 25$. So Quantity B is greater than Quantity A, choice B.

Notes: (1) The area of a square is $A = s^2$ where s is the length of a side of the square.

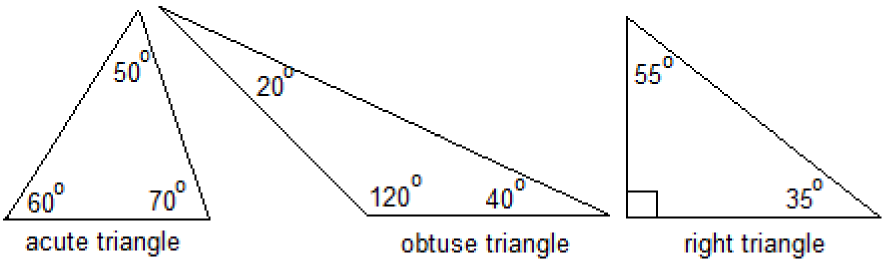
In square $ABCD$ we are given that $A = 25$. So $s^2 = 25$ and it follows that $s = \sqrt{25} = 5$.

(2) Normally the equation $s^2 = 25$ would have the two solutions $s = \pm 5$. But a length cannot be negative and so we consider only the positive solution.

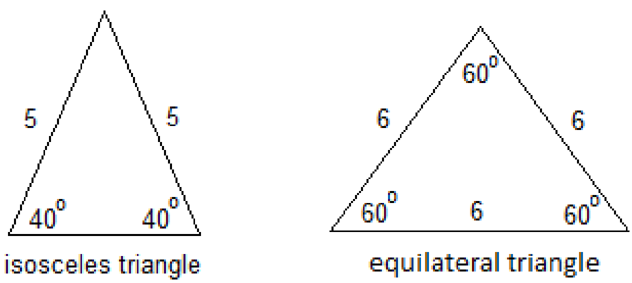
(3) An equilateral triangle has 3 sides of equal length. Equivalently, an equilateral triangle has 3 angles of equal measure (in which case they all measure $60°$) .

(4) To get the perimeter of a geometric figure we add up all the side lengths. Note that CD is <u>not</u> part of the perimeter of $ADECB$.

<u>A basic lesson in triangles:</u> A **triangle** is a two-dimensional geometric figure with three sides and three angles. The sum of the degree measures of all three angles of a triangle is 180.

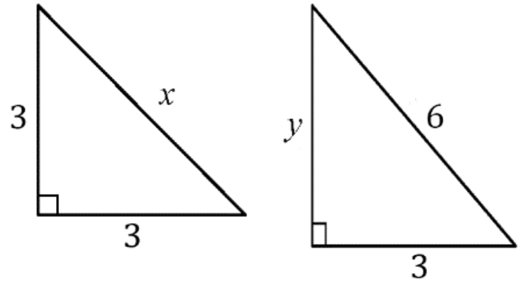


A triangle is **acute** if all three of its angles measure less than 90 degrees. A triangle is **obtuse** if one angle has a measure greater than 90 degrees. A triangle is **right** if it has one angle that measures exactly 90 degrees.



A triangle is **isosceles** if it has two sides of equal length. Equivalently, an isosceles triangle has two angles of equal measure.

A triangle is **equilateral** if all three of its sides have equal length. Equivalently, an equilateral triangle has three angles of equal measure (all three angles measure 60 degrees).



20. Quantity A: x
    Quantity B: y

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

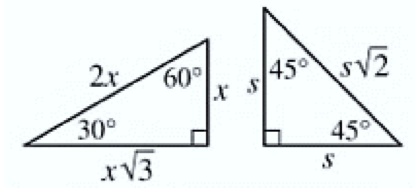
*** Solution using the Pythagorean theorem:** By the Pythagorean Theorem we have $x^2 = 3^2 + 3^2 = 9 + 9 = 18$. Also by the Pythagorean Theorem we have $y^2 = 6^2 - 3^2 = 36 - 9 = 27$. Since $y^2 > x^2$, and x and y are both positive, $y > x$. So the answer is B.

Notes: (1) The Pythagorean Theorem says that $c^2 = a^2 + b^2$ where a and b are the lengths of the legs of the right triangle and c is the length of the hypotenuse.

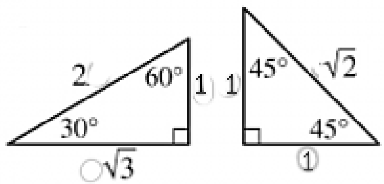
(2) When applying the Pythagorean Theorem always remember that the hypotenuse is by itself on one side of the equation. So for the rightmost triangle we have $6^2 = y^2 + 3^2$, or equivalently $y^2 = 6^2 - 3^2$.

(3) The leftmost triangle is an **isosceles right triangle**. It is isosceles because the two legs have the same length. An isosceles right triangle is the same as a 45, 45, 90 triangle, and so the hypotenuse has length $x = 3\sqrt{2}$. This could be rewritten as $3\sqrt{2} = \sqrt{9}\sqrt{2} = \sqrt{9 \cdot 2} = \sqrt{18}$ to make it easier to compare to y.

(4) For the GRE, it is useful to know the following two special triangles:



Some students get a bit confused because there are variables in these pictures. We can simplify the pictures if we substitute a 1 in for the variables.



Notice that the sides of the 30, 60, 90 triangle are then 1, 2 and $\sqrt{3}$ and the sides of the 45, 45, 90 triangle are 1, 1 and $\sqrt{2}$. The variables in the first picture above just tell us that if we multiply one of the sides in the second picture by a number, then we have to multiply the other two sides by the same number. For example, instead of 1, 1 and $\sqrt{2}$, we can have 3, 3 and $3\sqrt{2}$ (here $s = 3$), or $\sqrt{2}$, $\sqrt{2}$, and 2 (here $s = \sqrt{2}$).

21. If the degree measures of the three angles of a triangle are 50°, z°, and z°, what is the value of z ?

A. 60
B. 65
C. 70
D. 75
E. 80

Solution by starting with choice C: Recall that a triangle has angle measures that sum to $180°$. Let us start with choice C and guess $z = 70$. Then the angle measures sum to $50 + z + z = 50 + 70 + 70 = 190$ degrees, a bit too large. We can therefore eliminate choices C, D, and E.

Let us try choice B next. So we are guessing that $z = 65$. It follows that the angle measures sum to $50 + z + z = 50 + 65 + 65 = 180$ degrees. Since this is correct, the answer is choice B.

*** Algebraic solution:** Using the fact that the angle measures of a triangle sum to 180° we have

$$50 + z + z = 180$$
$$50 + 2z = 180$$
$$2z = 130$$
$$z = 65$$

Therefore, the answer is choice B.

22. The volume of a rectangular box is 2 cubic inches. If the width of the box is 4 inches and the height is $\frac{1}{4}$ inch, what is the length?

A. $\frac{1}{8}$ inch
B. $\frac{1}{4}$ inch
C. 1 inch
D. 2 inches
E. 4 inch

* The formula for the volume of a box is $V = \ell wh$. So we have

$$V = \ell wh$$
$$2 = \ell(4)(\tfrac{1}{4})$$
$$2 = \ell$$

Therefore, the answer is choice D.

Note: This problem can also be solved by starting with choice C. I leave the details of this solution to the reader.

23. In triangle CAT, the measure of angle C is 45° and the measure of angle A is between 10° and 40°. Which of the following could be the measure of angle T ? Indicate <u>all</u> such measures.

A. 80°
B. 90°
C. 100°
D. 110°
E. 120°
F. 130°
G. 140°

* The sum of the measures of angles C and A is between $45+10=55°$ and $45+40=85°$. It follows that the measure of angle T must be between $180-85=95°$ and $180-55=125°$. So the answers are choices C, D, and E.

Note: It is not entirely clear if the extreme values of $95°$ and $125°$ should be included as solutions. This is due to the ambiguity of the word "between."

In this question it doesn't matter because neither of those numbers appear as answer choices.

See problem 59 for more about this.

24. In the xy-plane, points A and B have coordinates $(3,-2)$ and $(-5,4)$, respectively. If point C is the midpoint of line segment AB, and C has coordinates (x,y), then what is the value of xy ?

* $(x,y)=\left(\frac{3-5}{2},\frac{-2+4}{2}\right)=\left(\frac{-2}{2},\frac{2}{2}\right)=(-1,1)$. So $xy=(-1)(1)=\mathbf{-1}$.

Note: We find the x-coordinate of the midpoint of AB by taking the average of the x-coordinates of A and B.

In this problem the x-coordinate of point A is 3 and the x-coordinate of point B is -5. It follows that the x-coordinate of the midpoint of AB is $\frac{3+(-5)}{2}=\frac{-2}{2}=-1$.

Similarly, we find the y-coordinate of the midpoint of AB by taking the average of the y-coordinates of A and B.

In this problem the y-coordinate of point A is -2 and the y-coordinate of point B is 4. It follows that the y-coordinate of the midpoint of AB is $\frac{-2+4}{2}=\frac{2}{2}=1$.

LEVEL 1: DATA ANALYSIS

A menu lists 6 meals and 5 drinks.

25. Quantity A: The number of different meal-drink combinations
    Quantity B: 11

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* We will use the **counting principle** which says that if one event is followed by a second independent event, the number of possibilities is multiplied. So in this example, the number of meal-drink combinations is $6 \cdot 5 = 30$. Since 30 is greater than 11, the answer is A.

Remark: The 2 events here are "choosing a meal," and "choosing a drink."

Creating a list: If you are having trouble understanding why we multiply in this problem, try writing out your own list and it should become clear. So, for example, suppose that our six meal choices are chicken, beef, fish, pasta, salad, and soup. Suppose our five drink choices are water, juice, soda, coffee, and tea. Here is a beginning of the list of meal-drink combinations. See if you can finish this list:

Chicken and water
Chicken and juice
Chicken and soda
Chicken and coffee
Chicken and tea
Beef and water
Beef and juice……….

26. Quantity A: The average (arithmetic mean) of 26, 53, and 125
    Quantity B: The average (arithmetic mean) of 25, 54, and 125

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Quick solution:** Both Quantities have the same sum, and therefore the same average. So the answer is C.

Note: One of the numbers mentioned in Quantity B is 1 less than a number mentioned in Quantity A, and another number mentioned in Quantity B is 1 more than a different number mentioned in Quantity A. The third number is the same in each Quantity. It follows that the 3 numbers mentioned in Quantity A have the same sum as the 3 numbers mentioned in Quantity B. Since both Quantities have the same sum, and the same amount of numbers, the two averages are the same.

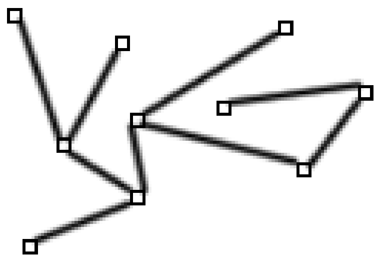
Computational solution: $\frac{26+53+125}{3}=\frac{204}{3}=68$, $\frac{25+54+125}{3}=\frac{204}{3}=68$, and therefore both Quantities are equal. So the answer is C.

Definition: The **average (arithmetic mean)** of a list of numbers is the sum of the numbers in the list divided by the quantity of the numbers in the list.

$$\textbf{Average} = \frac{\textbf{Sum}}{\textbf{Number}}$$

In GRE problems we sometimes use the formula in the following form:

$$\textbf{Sum} = \textbf{Average} \cdot \textbf{Number}$$



Each □ represents a nail and each □—□ represents a wire.

27. Quantity A: The number of nails
    Quantity B: The number of wires

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by counting:** There are 10 nails (represented by) and there are 9 wires (represented by). So Quantity A is greater than Quantity B, choice A.

$$\text{List } A\text{: } 35, 14, 63, 22, 53, 35$$

28. Quantity A: The median of the numbers in list *A*
    Quantity B: The mode of the numbers in list *A*

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* Let's list the numbers in increasing order:

$$14, 22, 35, 35, 53, 63$$

It should now be easy to see that the median and mode are both 35, choice C.

Definitions: The **median** of a list of numbers is the middle number when the numbers are arranged in increasing order. If the total number of values in the list is even, then the median is the average of the two middle values.

The **mode** of a list of numbers is the number that occurs most frequently. There can be more than one mode if more than one number occurs with the greatest frequency.

29. The average (arithmetic mean) of three numbers is 100. If two of the numbers are 80 and 130, what is the third number?

    A. 70
    B. 80
    C. 90
    D. 100
    E. 110

*** Solution by changing the average to a sum:** We change the average to a sum using the formula

Sum = Average · Number

We are averaging 3 numbers so that the Number is 3. The Average is given to be 100. Therefore the Sum of the 3 numbers is $100 \cdot 3 = 300$. Since we know that two of the numbers are 80 and 130, the third number is $300 - 80 - 130 = 90$, choice C.

$$\text{List } A\text{: } 58, 35, 72, 46, 49$$
$$\text{List } B\text{: } 70, 53, 11, 20, 68$$

30. The median of the numbers in list B is how much greater than the median of the numbers in list A ?

   A. 2
   B. 4
   C. 7
   D. 7.6
   E. 8

***** Let's rearrange the numbers in each list so that they appear in increasing order:

$$\text{List } A\text{: } 35, 46, \mathbf{49}, 58, 72$$
$$\text{List } B\text{: } 11, 20, \mathbf{53}, 68, 70$$

The median of the numbers in list B is 53 and the median of the numbers in list A is 49. So the answer is $53 - 49 = 4$, choice B.

31. For which of the following lists of 5 numbers is the average (arithmetic mean) less than the median? Indicate <u>all</u> such lists.

   A. $1, 1, 3, 4, 4$
   B. $1, 2, 3, 5, 6$
   C. $1, 1, 3, 5, 5$
   D. $1, 2, 3, 4, 5$
   E. $1, 2, 3, 4, 4$
   F. $1, 2, 3, 4, 9$

All of these lists have a median of 3.

It is very often easiest to work with the **Sum** of the numbers instead of the Average. We can easily change an average to a sum using the formula

Sum = Average · Number

In this case we want the sum to be less than $3 \cdot 5 = 15$. We check each choice:

A: $1+1+3+4+4=\mathbf{13}$
B: $1+2+3+5+6=17$
C: $1+1+3+5+5=15$
D: $1+2+3+4+5=15$
E: $1+2+3+4+4=\mathbf{14}$
F: $1+2+3+4+9=19$

The answers are choices A and E.

*** Quick Solution:** With a little experience it is not hard to see that A is an answer. Just look at how the numbers are "balanced" about the middle number 3. 1 is two units to the left, and 4 is only 1 unit to the right.

A similar argument can be used to see that choice E is an answer, and none of the other choices are answers.

So the answers are choices A and E.

$$9, 9, 9, 7, 7, 15, 11, x$$

32. The eight numbers shown represent the weight, in pounds, of eight cats in a pet store. What is the median weight, in pounds, of the eight cats in the pet store?

* Let's list all of the numbers (except x) in increasing order:

$$7, 7, 9, 9, 9, 11, 15$$

Now simply note that no matter where x is, there will always be two 9's in the middle. So the answer is **9**.

LEVEL 2: ARITHMETIC

A certain recipe requires 3 tablespoons of oil and makes 3 dozen brownies. (1 dozen = 12)

33. Quantity A: The amount of oil required for the same recipe to make 15 brownies

    Quantity B: 1.5 tablespoons

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

Solution by setting up a ratio: We begin by identifying 2 key words. In this case, such a pair of key words is "brownies" and "oil." We now compute Quantity A by setting up a ratio.

brownies	36	15
oil	3	x

Choose the words that are most helpful to you. Notice that we wrote in the number of brownies next to the word brownies, and the amount of oil (in tablespoons) next to the word oil. Also notice that the amount of oil required to make 36 brownies is written under the 36, and the (unknown) amount of oil needed to make 15 brownies is written under the 15. Now draw in the division symbols and equal sign, cross multiply and divide the corresponding ratio to find the unknown quantity x.

$$\frac{36}{3} = \frac{15}{x}$$
$$36x = 3 \cdot 15$$
$$36x = 45$$
$$x = \frac{45}{36} = 1.25$$

So the amount of oil required to make 12 brownies is 1.25 tablespoons. Since 1.5 is greater than this number, the answer is choice B.

Note: Since 1 dozen $= 12$, we have that 3 dozen $= 12 \cdot 3 = 36$. This is why we used the number 36 in the above solution.

*** Mental math:** Since 3 tablespoons of oil is required for $12 \cdot 3 = 36$ brownies, 1.5 tablespoons of oil is required for $\frac{36}{2} = 18$ brownies. It follows that 15 brownies requires less than 1.5 tablespoons of oil, and therefore Quantity B is greater, choice B.

$$x = 6 \text{ and } 4 < y < 5$$

34. Quantity A: $\frac{y}{x}$

Quantity B: 0.82

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If we let $y = 4.1$, we get $\frac{y}{x} = \frac{4.1}{6} \approx 0.68$ which is less than 0.82. If we let $y = 4.99$ we get $\frac{y}{x} = \frac{4.99}{6} \approx 0.83$ which is greater than 0.82. So the answer is D.

Note: Since $4 < y < 5$, we have $\frac{4}{6} < \frac{y}{x} < \frac{5}{6}$, or $0.\overline{6} < \frac{y}{x} < 0.8\overline{3}$. Since 0.82 is between $0.\overline{6}$ and $0.8\overline{3}$, it seems likely that the answer is D.

p_1 and p_2 are distinct prime numbers greater than 10, q_1 is the greatest prime number less than p_1, and q_2 the greatest prime number less than p_2

35. Quantity A: $p_1 - q_1$
    Quantity B: $p_2 - q_2$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

Solution by picking numbers: Let's let $p_1 = 11$ and $p_2 = 13$. Then we have $q_1 = 7$ and $q_2 = 11$. It follows that $p_1 - q_1 = 11 - 7 = 4$ and $p_2 - q_2 = 13 - 11 = 2$. So Quantity A is greater.

If we let $p_1 = 13$ and $p_2 = 11$, then we have $q_1 = 11$ and $q_2 = 7$. It follows that $p_1 - q_1 = 13 - 11 = 2$ and $p_2 - q_2 = 11 - 7 = 4$. So this time Quantity B is greater. So the answer is D.

*** Quick solution:** p_1 and p_2 are interchangeable in this problem. So the answer is most likely D (although C is still a possibility). To rule out C we need only choose p_1 and p_2 so that $p_1 - q_1$ and $p_2 - q_2$ are different. Letting $p_1 = 11$ and $p_2 = 13$, for example, will give the desired result (see the first paragraph of the previous solution for details).

$x > 0$, $y > 0$, and 7 percent of x is equal to 11 percent of y

36. Quantity A: x
    Quantity B: y

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Algebraic solution:** We are given that $0.07x = 0.11y$. So we have that $x = \frac{0.11}{0.07}y \approx 1.57y > y$. So Quantity A is greater, choice A.

Some notes on converting between decimal and percent:

(1) To change a percent to a decimal, divide by 100, or equivalently move the decimal point two places to the left (adding zeros if necessary).

For example, $7\% = .07$ and $11\% = .11$.

(2) To change a decimal to a percent, multiply by 100, or equivalently move the decimal point two places to the right (adding zeros if necessary).

37. What is the greatest positive integer that is a divisor of 10, 25, and 45?

A. 1
B. 2
C. 3
D. 5
E. 10

Solution by starting with choice E (using the calculator): Since the question has the word **"greatest"** in it, we will start with the greatest answer choice which is choice E, and we will divide each of the three numbers by 10. Since 25 divided by 10 is 2.5 (not an integer), choice E is not the answer. We next try choice D. The divisions by 5 give us 2, 5 and 9 respectively. Since these are all integers, the answer is choice D.

Note: The three given integers are all divisible by 1, but choice A is not the answer because 5 is greater.

*** Solution by starting with choice E (without the calculator):** As in the last solution, we begin with the greatest answer choice. Since 25 (as well as 45) does not end in a 0, it is not divisible by 10. Since all three integers end in a 0 or a 5, they are all divisible by 5. Thus, the answer is choice D.

Direct solution: We are being asked to find the **greatest common factor** (or **gcf**) of 10, 25 and 45, which is 5, choice D.

Finding the greatest common factor: Here are two ways to find the greatest common factor of the given integers.

(1) List all factors of each integer and look for the biggest one they have in common.

Factors of 10: $\{\mathbf{1}, 2, \mathbf{5}, 10\}$
Factors of 25: $\{\mathbf{1}, \mathbf{5}, 25\}$
Factors of 45: $\{\mathbf{1}, 3, \mathbf{5}, 9, 15, 45\}$

Common Factors: $\{1, 5\}$

Thus, the greatest common factor is 5.

(2) Here is a more sophisticated method that is much quicker if the given integers are large.

<u>Step 1</u>: Find the prime factorization of each number in the set.

$$10 = 2 \cdot 5$$
$$25 = 5^2$$
$$45 = 3^2 \cdot 5$$

<u>Step 2</u>: Choose the lowest power of each prime that appears in **all** of the factorizations. In this case, this is just 5.

<u>Step 3</u>: Multiply these numbers together to get the greatest common factor. (In this case there is nothing to multiply since there is only one prime factor that the three integers have in common.)

Remark: We can also write the above prime factorizations as follows:

$$10 = 2^1 \cdot 3^0 \cdot 5^1$$
$$25 = 2^0 \cdot 3^0 \cdot 5^2$$
$$45 = 2^0 \cdot 3^2 \cdot 5^1$$

It is easy to see in this form that the lowest power of 2 is $2^0 = 1$, and similarly the lowest power of 3 is 3^0.

Definitions: An integer n is **divisible** by an integer d if there is another integer k such that $n = dk$. For example, 42 is divisible by 7 because $42 = 7 \cdot 6$. In practice we can check if n is divisible by d simply by dividing n by d in the calculator. If the answer is an integer, then n is divisible by d. If the answer is not an integer (it contains digits after the decimal point), then n is not divisible by d. If n is divisible by d, we say that d is a **divisor** (or **factor**) of n.

The **greatest common factor (gcf)** of a set of positive integers is the largest positive integer that each integer in the set is divisible by.

The greatest common factor is also known as the **greatest common divisor (gcd)**.

The Fundamental Theorem of Arithmetic: Every integer greater than 1 can be written "uniquely" as a product of primes.

The word "uniquely" is written in quotes because prime factorizations are only unique if we agree to write the primes in increasing order.

For example, 6 can be written as $2 \cdot 3$ or as $3 \cdot 2$. But these two factorizations are the same except that we changed the order of the factors. To make things as simple as possible we always agree to use the **canonical representation**. The word "canonical" is just a fancy name for "natural," and the most natural way to write a prime factorization is in increasing order of primes. So the canonical representation of 6 is $2 \cdot 3$.

As another example, the canonical representation of 18 is $2 \cdot 3 \cdot 3$. We can tidy this up a bit by rewriting $3 \cdot 3$ as 3^2. So the canonical representation of 18 is $2 \cdot 3^2$.

If you are new to factoring, you may find it helpful to draw a factor tree. For example, here is a factor tree for 18:

$$\begin{array}{ccccc} & 18 & & & \\ \swarrow & & \searrow & & \\ \boxed{2} & & 9 & & \\ & \swarrow & & \searrow & \\ & \boxed{3} & & \boxed{3} & \end{array}$$

To draw this tree we started by writing 18 as the product $2 \cdot 9$. We put a box around 2 because 2 is prime, and does not need to be factored anymore. We then proceeded to factor 9 as $3 \cdot 3$. We put a box around each 3 because 3 is prime. We now see that we are done, and the prime factorization can be found by multiplying all of the boxed numbers together. Remember that we will usually want the canonical representation, so write the final product in increasing order of primes.

By the Fundamental Theorem of Arithmetic above it does not matter how we factor the number – we will always get the same canonical form. For example, here is a different factor tree for 18:

$$18$$
$$\swarrow\searrow$$
$$\boxed{3} \quad 6$$
$$\swarrow\searrow$$
$$\boxed{2} \quad \boxed{3}$$

38. If m and n are integers and $m+n$ is odd, which of the following must be odd?

A. m
B. mn
C. $2m+n$
D. $3m-n$
E. $mn+m$

Solution by picking numbers: Let's choose values for m and n satisfying the given condition, say $m=2, n=3$. Note that $m+n=5$ which is odd.

We now substitute these values for m and n into each answer choice.

A.	2	even
B.	6	even
C.	7	odd
D.	3	odd
E.	8	even

Since A, B, and E came out even we can eliminate them.

Let's now try $m=3$, $n=2$. Once again note that $m+n=5$ which is odd.

We now substitute these values for m and n into choices C and D.

C.	8	even
D.	7	odd

Since C came out even we can eliminate it. The answer is therefore D.

Important note: It is very important that you check every answer choice when picking numbers. As we have seen in this problem specific numbers can lead to more than one choice coming out correct.

*** Direct solution:** In order for $m+n$ to be odd, either m is even and n odd, or m is odd and n even.

If m is even and n odd, then $3m$ is even, and so $3m - n$ is odd.

$$[\,o \cdot e - o = e - o = o]$$

If m is odd and n even, then $3m$ is odd, and so $3m - n$ is odd.

$$[\,o \cdot o - e = o - e = o]$$

Since both cases give a result that is odd, the answer is D.

Notes: (1) Two integers are said to have the same **parity** if they are both even or both odd.

For example, 4 and 10 have the same parity, whereas 4 and 5 have different parities.

(2) Parity has nothing to do with whether the integer is positive or negative. For example, -5 and 3 have the same parity because they are both odd.

(3) When we add two integers with the same parity, the resulting integer is even.

When we add two integers with different parities, the resulting integer is odd.

For example, $4 + 10 = 14$ and $-5 + 3 = -2$. Both of these results are even because we are adding two integers with the same parity.

Also, $4 + 5 = 9$. This result is odd because we are adding integers with different parities.

(4) The product of an even integer and any other integer is even, whereas the product of two odd integers is odd.

For example, $4 \cdot 5 = 20$. The result is even because 4 is even.

Also, $3 \cdot 5 = 15$. The result is odd because both integers being multiplied are odd.

(5) We can summarize the results in the previous notes as follows (e stands for "even" and o stands for "odd"):

$$e + e = e \qquad e + o = o \qquad o + e = o \qquad o + o = e$$
$$e \cdot e = e \qquad e \cdot o = e \qquad o \cdot e = e \qquad o \cdot o = o$$

(6) Subtraction behaves the same as addition. For example, $e - o = o$.

(7) As an example, suppose that m is odd and n is even. Then we have

$$2m + n = e \cdot o + e = e + e = e$$

This computation eliminates choice C.

See if you can eliminate choices A, B, and E this way as well.

39. Last month Jacob spent between $\frac{1}{5}$ and $\frac{1}{4}$ of his net income to fix his car. If Jacob spent \$875 to fix his car last month, which of the following could have been his net income last month? Indicate <u>all</u> such net incomes.

A. \$3450
B. \$3550
C. \$4250
D. \$4550
E. \$5450

***** $875 \cdot 4 = 3500$ and $875 \cdot 5 = 4375$. So the answers are B and C.

Notes: (1) If we let x represent Jacob's net income, then we are given that $\frac{1}{5}x < 875$ and $875 < \frac{1}{4}x$.

To solve the first inequality, we multiply each side of the inequality by 5 to get $x < 875 \cdot 5 = 4375$.

To solve the second inequality, we multiply each side of the inequality by 4 to get $875 \cdot 4 < x$, or equivalently, $3500 < x$.

Putting the two inequalities together yields $3500 < x < 4375$.

(2) It is not entirely clear if the extreme values of 3500 and 4375 should be included as solutions. This is due to the ambiguity of the word "between."

In this question it doesn't matter because neither of those numbers appear as answer choices.

See problem 59 for more about this.

40. Three consecutive integers are listed in increasing order. If their sum is 732, what is the second integer in the list?

Solution by guessing: Let us try some guesses for the second integer.

2nd integer	1st integer	3rd integer	Sum
200	199	201	600
250	249	251	750
240	239	241	720
245	244	246	735
244	243	245	732

Thus, the answer is $\mathbf{244}$.

Remark: You should use the calculator to compute these sums. This will be quicker and you are less likely to make a careless error.

Algebraic solution: If we name the least integer x, then the second and third integers are $x + 1$ and $x + 2$, respectively. So we have

$$x + (x + 1) + (x + 2) = 732$$
$$3x + 3 = 732$$
$$3x = 729$$
$$x = 243$$

The second integer is $x + 1 = \mathbf{244}$.

*** Quick solution:** Simply divide 732 by 3 to get $\mathbf{244}$.

Remark for the advanced student: The following algebraic steps show why the advanced method gives the correct solution.

$$x + (x + 1) + (x + 2) = 732$$
$$3x + 3 = 732$$
$$3(x + 1) = 732$$
$$x + 1 = 244$$

Note that the last two steps show that $x + 1 = \frac{732}{3}$.

Definition: Consecutive integers are integers that follow each other in order. The difference between consecutive integers is 1. Here are two examples.

$1, 2, 3$	these are three consecutive integers
$-3, -2, -1, 0, 1$	these are five consecutive integers

In general, if x is an integer, then $x, x + 1, x + 2, x + 3, \dots$ are consecutive integers.

LEVEL 2: ALGEBRA

$$(x - 3y)(x + 3y) = 7$$

41. Quantity A: $x^2 - 9y^2$
    Quantity B: 7

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution using a special factoring formula:** We use the formula for the difference of two squares:

$$(x - 3y)(x + 3y) = x^2 - 9y^2$$

So the two Quantities are equal, choice C.

Notes: (1) The **difference of two squares** can be factored as follows:

$$a^2 - b^2 = (a - b)(a + b)$$

In this problem $a = x$, $b = 3y$, and we are applying the formula backwards.

$$(x - 3y)(x + 3y) = x^2 - (3y)^2$$

(2) $(3y)^2 = (3y)(3y) = 3 \cdot 3 \cdot y \cdot y = 9y^2$

Solution using FOIL:

$$(x - 3y)(x + 3y) = x^2 + 3xy - 3xy - 9y^2 = x^2 - 9y^2$$

So the two Quantities are equal, choice C.

Solution using the standard algorithm for multiplication:

$$\begin{array}{r} x - 3y \\ \underline{x + 3y} \\ 3xy - 9y^2 \\ \underline{x^2 - 3xy } \\ x^2 + 0xy - 9y^2 \end{array}$$

So $(x-3y)(x+3y) = x^2 + 0xy - 9y^2 = x^2 - 9y^2$, and we see that the two Quantities are equal, choice C.

The algorithm step by step: We begin by lining up the polynomials vertically:

$$\begin{array}{r} x - 3y \\ \underline{x + 3y} \end{array}$$

We multiply the $3y$ on the bottom by each term on top, moving from right to left. First note that $3y$ times $-3y$ is $-9y^2$:

$$\begin{array}{r} x - 3y \\ \underline{x + 3y} \\ -9y^2 \end{array}$$

Next note that $3y$ times x is $3xy$:

$$\begin{array}{r} x - 3y \\ \underline{\quad x + 3y} \\ 3xy - 9y^2 \end{array}$$

Now we multiply the x on the bottom by each term on top, moving from right to left. This time as we write the answers we leave one blank space on the right:

$$\begin{array}{rr} & x - 3y \\ & \underline{x + 3y} \\ & 3xy - 9y^2 \\ \underline{x^2 - 3xy} & \end{array}$$

Finally, we add:

$$\begin{array}{rr} & x - 3y \\ & \underline{x + 3y} \\ & 3xy - 9y^2 \\ \underline{x^2 - 3xy} & \\ x^2 + 0xy - 9y^2 & \end{array}$$

$$0 < a < b$$

42. Quantity A: $\frac{a}{5}$

Quantity B: $\frac{b}{4}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Algebraic solution:** Dividing each side of the inequality $a < b$ by 5 gives

$$\frac{a}{5} < \frac{b}{5}.$$

Since $5 > 4$ and $b > 0$, we have $\frac{b}{5} < \frac{b}{4}$. Therefore,

$$\frac{a}{5} < \frac{b}{5} < \frac{b}{4}.$$

That is, Quantity B is greater than Quantity A, choice B.

Notes: (1) We can plug in values for a and b to eliminate answer choices. For example, if we let $a = 1$ and $b = 2$ (note that $a < b$), then we have $\frac{a}{5} = \frac{1}{5} = 0.2$ and $\frac{b}{4} = \frac{1}{4} = 0.25$. So Quantity B is greater.

This eliminates choices A and C (or equivalently, narrows down our choices to B or D).

(2) We can continue to try additional values for a and b with $0 < a < b$. Each time we do this we will get more evidence that choice B is the answer (because $\frac{a}{5}$ will always be less than $\frac{b}{4}$). But we could never be absolutely certain that choice D is not the answer.

(3) To increase our chances of being successful using this method we should try different "types" of values for a and b. For example, we just tried two small positive integers. Maybe we also want to try two fractions, a fraction and an integer, and so on.

(4) Increasing the denominator of a positive fraction *decreases* the size of the number. For example, since $5 > 4$, we have $\frac{1}{5} < \frac{1}{4}$. Since $b > 0$, we also have $\frac{b}{5} < \frac{b}{4}$.

(5) If b were negative, then multiplying each side of an inequality by b reverses the inequality (so if $b < 0$, then $5 > 4 \Rightarrow \frac{1}{5} < \frac{1}{4} \Rightarrow \frac{b}{5} > \frac{b}{4}$).

(6) If we change the condition $0 < a < b$ to $a < b$, we get a much harder problem (See Supplemental Problem 105).

$$k > 1$$

43. Quantity A: $(k^4)^4$
    Quantity B: $k(k^3)^5$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Algebraic solution:** We have

$$(k^4)^4 = k^{4\cdot 4} = k^{16}$$

and

$$k(k^3)^5 = k(k^{3\cdot 5}) = k(k^{15}) = k^1\, k^{15} = k^{1+15} = k^{16}$$

So the two Quantities are equal, choice C.

Note: As in problem 42, we can plug in a value for k that is greater than 1 and use the calculator to narrow down our choices to C or D. Several attempts for values of k would lead us to guess that C was the answer. For example, trying $k = 2$, $k = 1.1$, and $k = 4.3$ should give sufficient evidence.

Laws of Exponents: For those students that have forgotten, here is a brief review of the laws of exponents needed for the GRE:

Law	Example
$x^0 = 1$	$3^0 = 1$
$x^1 = x$	$9^1 = 9$
$x^a x^b = x^{a+b}$	$x^3 x^5 = x^8$
$x^a/x^b = x^{a-b}$	$x^{11}/x^4 = x^7$
$(x^a)^b = x^{ab}$	$(x^5)^3 = x^{15}$
$(xy)^a = x^a y^a$	$(xy)^4 = x^4 y^4$
$(x/y)^a = x^a/y^a$	$(x/y)^6 = x^6/y^6$
$x^{-1} = 1/x$	$3^{-1} = 1/3$
$x^{-a} = 1/x^a$	$9^{-2} = 1/81$
$x^{1/n} = \sqrt[n]{x}$	$x^{1/3} = \sqrt[3]{x}$
$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{9/2} = \sqrt{x^9} = \left(\sqrt{x}\right)^9$

$$3x - 5y < 12$$

44. Quantity A: $6x - 10y$
    Quantity B: 23

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Algebraic solution:** We multiply each side of the given inequality by 2 to get $2(3x - 5y) < 2 \cdot 12$, or equivalently $6x - 10y < 24$. So it seems reasonable that we can make the value of $6x - 10y$ less than or greater than 23 (see the next solution for details), choice D.

Note: Quantity A can be factored as $6x - 10y = 2(3x - 5y)$. This is why we should know to multiply each side of the given inequality by 2.

Solution by picking numbers: Let's try $x = 0$ and $y = 0$. Then we have $3x - 5y = 0 < 12$ so that the given condition is satisfied. We also have that $6x - 10y = 0$. So Quantity B is greater than Quantity A, and we have narrowed the answer down to choice B or D.

Now let's try something that makes the left hand side of the inequality a number very close to 12, let's say $x = 3.99$ and $y = 0$. Note that in this case $3x - 5y = 3 \cdot 3.99 = 11.97 < 12$ so that the condition is satisfied.

We also have that $6x - 10y = 6 \cdot 3.99 = 23.94 > 23$. So Quantity A is greater. This eliminates choice B, and so the answer is D.

$$15x + 6y = 31$$
$$6x + 24y = 17$$

45. If x and y satisfy the system of equations given above, what is the value of $x - 2y$?

    A. $\frac{9}{14}$
    B. $\frac{14}{9}$
    C. 7
    D. 10
    E. 14

*** Solution by trying a simple operation:** We subtract the second equation from the first:

$$\begin{array}{r} 15x + 6y = 31 \\ \underline{6x + 24y = 17} \\ 9x - 18y = 14 \end{array}$$

We divide each side of the equation by 9 to get $x - 2y = \frac{14}{9}$, choice B.

Notes: (1) When we divide $9x - 18y$ by 9, we need to divide each term by 9. That is $\frac{9x-18y}{9} = \frac{9x}{9} - \frac{18y}{9} = x - 2y$.

(2) We can also factor 9 out from the left hand side and then divide as follows: $9x - 18y = 9(x - 2y)$.

So we have $9(x - 2y) = 14$, and therefore, $x - 2y = \frac{14}{9}$.

(3) We can also first solve the given system of equations by finding x and y, and then substituting these values into the expression $x - 2y$. This method is quite time consuming and so I omit it here and leave it as an optional exercise for the reader.

(4) Whenever we are trying to find an expression that involves addition, subtraction, or both, **adding or subtracting** the given equations usually does the trick.

46. The quantities x and y are positive and are related by the equation $xy = k$ where k is a constant. If the value of y increases by 25 percent, then the value of x decreases by what percent?

    A. 20%
    B. 25%
    C. $33\frac{1}{3}$%
    D. 75%
    E. 80%

*** Solution by picking numbers:** Let's start with $x = 10$ and $y = 10$ (and therefore $k = 10 \cdot 10 = 100$).

When we increase y by 25%, we get $y = 12.5$. It follows that

$$x = \frac{k}{y} = \frac{100}{12.5} = 8$$

and so x decreased by 20%, choice A.

Notes: (1) To increase $y = 10$ by 25% we multiply 10 by 1.25 to get

$$10 \cdot 1.25 = 12.5.$$

(2) To decrease 10 by 20%, we multiply 10 by $1 - .20 = .80$ to get

$$10 \cdot 0.8 = 8.$$

(3) Once we find that $x = 8$, we can find the percent decrease by using the following formula for percent change:

$$Percent\ Change = \frac{Change}{Original} \times 100$$

In this problem, the original value is 10 and the change is $10 - 8 = 2$. So we have $Percent\ Change = \frac{2}{10} \times 100 = 20\%$.

Direct solution: Solving the original equation for x gives us

$$x = \frac{k}{y}$$

When we increase y by 25% we get $y^* = 1.25y$. It follows that

$$x^* = \frac{k}{y^*} = \frac{k}{1.25y} = \frac{1}{1.25} \cdot \frac{k}{y} = .8\left(\frac{k}{y}\right) = .8x$$

We now use the percent change formula to get that the Percent Change is

$$PC = \frac{x - .8x}{x} \times 100 = \frac{x(1 - .8)}{x} \times 100 = .2 \times 100 = 20\%$$

This is choice A.

47. If $(x-5)(x+2)=0$, which of the following could be the value of $10x^{-3}$? Indicate all such values.

A. $-\frac{5}{4}$
B. $-\frac{2}{25}$
C. 0
D. $\frac{2}{25}$
E. $\frac{5}{4}$

*** Algebraic solution:** The equation $(x-5)(x+2)=0$ has the two solutions $x=5$ and $x=-2$.

If $x=5$, then $10x^{-3}=10(5)^{-3}=\frac{10}{5^3}=\frac{10}{125}=\frac{2}{25}$, choice D.

If $x=-2$, then $10x^{-3}=10(-2)^{-3}=\frac{10}{(-2)^3}=\frac{10}{-8}=-\frac{5}{4}$, choice A.

So the answers are choices A and D.

Notes: (1) The **zero property** says that if a and b are real numbers and $ab=0$, then $a=0$ or $b=0$.

In this problem we have $(x-5)(x+2)=0$. By the zero property we have $x-5=0$ or $x+2=0$. It follows that $x=5$ or $x=-2$.

(2) $x^{-n}=\frac{1}{x^n}$. In particular, $x^{-3}=\frac{1}{x^3}$.

(3) $(-2)^3=(-2)(-2)(-2)=4(-2)=-8$.

48. Printer A and printer B each work at a constant rate. Printer A can print 300 pages in 6 minutes and printer B can print 300 pages in 4 minutes. How many more copies can printer B make in 5 minutes than printer A can make in 7 minutes?

*** Quick solution:** Since printer A can print 300 pages in 6 minutes, printer A can print $\frac{300}{6}=50$ pages in 1 minute, and therefore $50\cdot 7=350$ pages in 7 minutes.

Since printer B can print 300 pages in 4 minutes, it can print $\frac{300}{4}=75$ pages in 1 minute, and therefore $75\cdot 5=375$ pages in 5 minutes.

So the answer is $375 - 350 = \mathbf{25}$.

Solution using ratios: We first set up a ratio for printer A:

pages	300	a
minutes	6	7

Now draw in the division symbols and equal sign, cross multiply and divide the corresponding ratio to find the unknown quantity a.

$$\frac{300}{6} = \frac{a}{7}$$
$$6a = 300 \cdot 7$$
$$6a = 2100$$
$$a = \frac{2100}{6} = 350$$

Next we set up a ratio for printer B:

pages	300	b
minutes	4	5

$$\frac{300}{4} = \frac{b}{5}$$
$$4b = 300 \cdot 5$$
$$4b = 1500$$
$$b = \frac{1500}{4} = 375$$

So the answer is $b - a = 375 - 350 = \mathbf{25}$.

LEVEL 2: GEOMETRY

49. Quantity A: The volume of a cylinder with a diameter of 6 inches and a height of 12 inches.

    Quantity B: The volume of a cylinder with a diameter of 12 inches and a height of 6 inches.

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* Quantity A is equal to $V = \pi r^2 h = \pi \cdot 3^2 \cdot 12 = 108\pi$ and Quantity B is equal to $V = \pi r^2 h = \pi \cdot 6^2 \cdot 6 = 216\pi$.

So Quantity B is greater than Quantity A, choice B.

Notes: (1) The volume of a cylinder is $\boldsymbol{V = \pi r^2 h}$, where r is the radius of a base of the cylinder and h is the height of the cylinder.

(2) The length of a diameter of a circle is twice the length of a radius of the circle. That is $\boldsymbol{d = 2r}$, or equivalently $\boldsymbol{r = \frac{d}{2}}$.

Definitions: A **circle** is a two-dimensional geometric figure formed of a curved line surrounding a center point, every point of the line being an equal distance from the center point. This distance is called the **radius** of the circle. The **diameter** of a circle is the distance between any two points on the circle that pass through the center of the circle.

A **cylinder** is a three-dimensional geometric solid bounded by two equal parallel circles and a curved surface formed by moving a straight line so that its ends lie on the circles.

The area of equilateral triangle A is 4 times the area of equilateral triangle B.

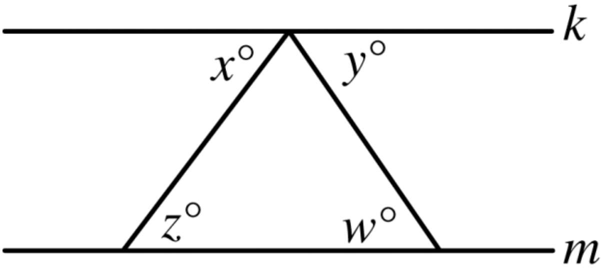
50. Quantity A: The ratio of the length of one side of A to the length of another side of A

    Quantity B: The ratio of the length of one side of B to the length of another side of B

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* In an equilateral triangle all side lengths are equal. It follows that the ratio of the length of one side of an equilateral triangle to another side of the same triangle is 1. So Quantities A and B are both equal to 1, and the answer is C.

Note: The given information about the areas of these triangles is not needed to solve the problem.



Line k is parallel to line m.

51. Quantity A: $x + w$
Quantity B: $y + z$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* k and m are parallel lines cut by two transversals.

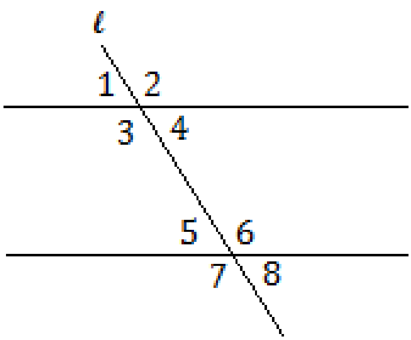
Using the leftmost transversal and alternate interior angles, we have that $x = z$.

Using the rightmost transversal and alternate interior angles, we have $y = w$.

It follows that $x + w = z + y = y + z$.

So Quantities A and B are equal, and the answer is C.

Notes: (1) The following picture shows two parallel lines cut by the transversal ℓ.



Angles 1, 4, 5, and 8 all have the same measure. Also, angles 2, 3, 6, and 7 all have the same measure. Any two angles that do not have the same measure are supplementary, that is their measures add to $180°$.

There are two pairs of **alternate interior angles**. These angle pairs are

3 and 6   4 and 5

(2) The figure in note (1) is similar to the figure given in the problem with just the rightmost transversal shown. Alternate interior angles 4 and 5 in this last figure correspond to angles y and w in the question.

A right circular cylinder with height 5 centimeters has volume 18π cubic centimeters.

52. Quantity A: The radius of a base of the cylinder
    Quantity B: $\sqrt{2}$ centimeters

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* To compute Quantity A we use the formula $V = \pi r^2 h$. In this question we are given that $h = 5$ and $V = 18\pi$. So we have $18\pi = \pi r^2 \cdot 5$.

It follows that $r^2 = \frac{18}{5}$, and so $r = \sqrt{\frac{18}{5}} = \sqrt{3.6} > \sqrt{2}$.

So Quantity A is greater than Quantity B, choice A.

Note: The volume of a cylinder is $\boldsymbol{V = \pi r^2 h}$, where r is the radius of a base of the cylinder and h is the height of the cylinder.

53. In the xy-plane, what is the slope of the line whose equation is $5x - 3y = 12$

    A. -4
    B. $-\frac{12}{5}$
    C. $\frac{3}{5}$
    D. $\frac{5}{3}$
    E. 3

Solution by putting the equation of the line into slope-intercept form: To find the slope of the line, we solve the given equation for y:

$$5x - 3y = 12$$
$$-3y = -5x + 12$$
$$y = \frac{5}{3}x - 4$$

So we see that the slope of the line is $\frac{5}{3}$, choice D.

Clarification of the algebra: To get from the first equation to the second equation we subtracted $5x$ from each side.

To get from the second equation to the third equation we divided each side of the equation by -3. Note that we had to divide each term on the right hand side by -3. So we have $\frac{-5x}{-3} = \frac{5x}{3} = \frac{5}{3}x$ and $\frac{12}{-3} = -4$.

Note: The equation of a line in **slope-intercept** form is

$$\boldsymbol{y = mx + b}$$

where m is the slope of the line and $(0, b)$ is the y-intercept of the line.

In the above solution we put the equation into slope-intercept form by solving for y.

*** Quick solution:** The slope of the line with equation in the **general form** $ax + by = c$ is $m = -\frac{a}{b}$. In this question, we have $a = 5$ and $b = -3$. So the slope is $m = -\frac{5}{-3} = \frac{5}{3}$, choice D.

Notes: (1) Memorizing this little fact about the slope of a line in general form is optional. If you do know it, then it's a nice quick way to get the slope of the line without having to do any algebra first. If you do not know it, then the first method of solution above is certainly acceptable.

(2) We can put the equation of a line in general form into slope-intercept form as follows:

$$ax + by = c$$
$$by = -ax + c$$
$$y = -\frac{a}{b}x + \frac{c}{b}$$

From this last equation we see that the slope is $m = -\frac{a}{b}$.

54. In an xy-coordinate system, which point lies in the interior of a circle with center (0,0) and radius 3 ?

A. $(1,-3)$
B. $(-1,-2)$
C. $(-3,1)$
D. $(0,3)$
E. $(3,3)$

*** Solution by process of elimination:** If one of the coordinates of the point is 3 or -3, then the point is either on or outside of the circle (if the other coordinate is 0 the point is on the circle, otherwise it is outside). We can therefore eliminate choices A, C, D and E. Thus, the answer is choice B.

Remark: A picture can help clarify the above solution if you are confused. Draw the given circle inscribed in a square. The points with a coordinate of 3 or -3 lie on this square.

Solution using the distance formula: In order for a point (x, y) to lie in the interior of a circle with center $(0,0)$ and radius 3, the distance between (x, y) and $(0,0)$ needs to be less than 3. We can use the distance formula to check this:

$$d^2 = (x-0)^2 + (y-0)^2 = x^2 + y^2$$

We want d to be less than 3, or equivalently d^2 to be less than 9.

Starting with choice C, we have $d^2 = (-3)^2 + 1^2 = 10$. This is too big.

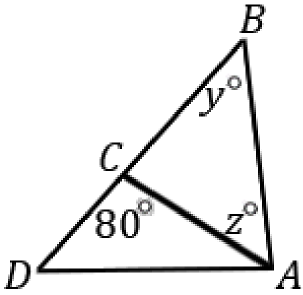
Let's try choice B. $d^2 = (-1)^2 + (-2)^2 = 5$. Since this is less than 9, choice B is the answer.

A common error: Many students get confused when squaring a negative number. Remember that squaring means "multiplying by itself." So for example, $(-3)^2 = (-3)(-3) = 9$. Compare this to the following computation: $-3^2 = (-1)(3^2) = (-1)(9) = -9$ (negating a number is equivalent to multiplying by -1). Notice that in this last computation we were careful to follow the usual order of operations. We performed the exponentiation before multiplying by -1.

Note: The distance between the two points (x, y) and (z, w) is given by

$$d = \sqrt{(z-x)^2 + (w-y)^2} \quad \text{or equivalently} \quad d^2 = (z-x)^2 + (w-y)^2$$

This formula is called the **distance formula**.



55. In ΔABDabove, if $25 < y < 45$, which of the following are possible values for z ? Indicate <u>all</u> such values.

A. 25
B. 30
C. 35
D. 40
E. 45
F. 50
G. 55
H. 60

Angles ACD and ACB form a linear pair and are therefore supplementary. So angle ACB has measure $180 - 80 = 100°$. Since the angle measures in a triangle add up to $180°$, it follows that

$$z = 180 - 100 - y = 80 - y.$$

We have $80 - 45 = 35$ and $80 - 25 = 55$. So $35 < z < 55$, and the answers are choices D, E, and F.

Notes: (1) We can solve for z more formally as follows:

$$25 < y < 45$$
$$-45 < -y < -25$$
$$80 - 45 < 80 - y < 80 - 25$$
$$35 < z < 55$$

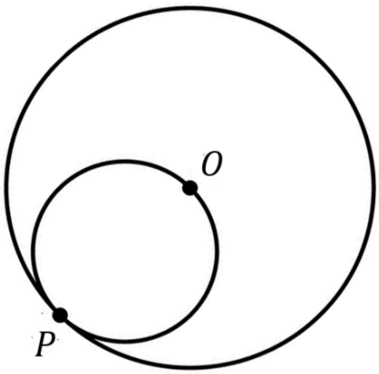
Observe that to get from the first inequality to the second inequality we multiplied each part by -1. Because we multiplied by a negative number, the inequalities were reversed. Some readers might find it useful to include the following additional step:

$$25 < y < 45$$
$$-25 > y > -45$$
$$-45 < -y < -25$$

*** Quicker solution:** We can get a relationship between y and z quickly by using the fact that **the measure of an exterior angle of a triangle is the sum of the measures of the two opposite interior angles of the triangle**.

In this problem we get $80 = y + z$, and therefore $z = 80 - y$.

As in the previous solution we have $80 - 45 = 35$ and $80 - 25 = 55$. So $35 < z < 55$, and the answers are choices D, E, and F.



56. In the figure above point O is the center of the larger circle and the two circles are tangent at point P. The line segment OP (not shown) is a diameter of the smaller circle. The area of the larger circle is what multiple of the area of the smaller circle?

*** Solution by picking a number:** Let's let the radius of the smaller circle be 1. Then the area of the smaller circle is $A = \pi r^2 = \pi(1)^2 = \pi$.

The radius of the larger circle is equal to the diameter of the smaller circle, which is 2. So the area of the larger circle is $A = \pi r^2 = \pi(2)^2 = 4\pi$.

So the area of the larger circle is 4 times the area of the smaller circle. In other words, the desired multiple is $\mathbf{4}$.

Geometric solution: Let r be the radius of the smaller circle. Then the area of the smaller circle is $A = \pi r^2$.

The radius of the larger circle is equal to the diameter of the smaller circle, which is $2r$. So the area of the larger circle is $A = \pi(2r)^2 = 4\pi r^2$.

So the area of the larger circle is 4 times the area of the smaller circle. In other words, the desired multiple is $\mathbf{4}$.

Notes: (1) The area of a circle is $\boldsymbol{A = \pi r^2}$, where r is the radius of the circle.

(2) The length of a diameter of a circle is twice the length of a radius of the circle. That is $\boldsymbol{d = 2r}$, or equivalently, $\boldsymbol{r = \frac{d}{2}}$.

(3) $(2r)^2 = (2r)(2r) = 4r^2$.

LEVEL 2: DATA ANALYSIS

List X: 28, 14, 63, 22, 53, 35

List Y (not shown) consists of 6 distinct numbers. Each number in list Y was obtained by dividing one of the numbers in list X by 7 and then subtracting 3 from the resulting number.

57. Quantity A: The range of the numbers in list Y
    Quantity B: 6

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* The smallest number in list X is 14. It follows that the smallest number in list Y is $\frac{14}{7} - 3 = 2 - 3 = -1$.

The greatest number in list X is 63. It follows that the greatest number in list Y is $\frac{63}{7} - 3 = 9 - 3 = 6$.

So the range of the numbers in list Y is $6 - (-1) = 6 + 1 = 7$.

Therefore, Quantity A is greater than Quantity B, choice A.

Definition: The **range** of a list of numbers is the positive difference between the greatest number and smallest number in the list.

The arithmetic mean of 300 measurements is 72, and the arithmetic mean of 200 additional measurements is 64.

58. Quantity A: The arithmetic mean of the 500 measurements
    Quantity B: 68

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Solution by changing averages to sums:** We change the averages to sums using the formula "**Sum = Average · Number.**"

First we are averaging 300 measurements so that the Number is 300. The Average is given to be 72. So the Sum of the 300 measurements is

$$72 \cdot 300 = 21{,}600.$$

Next we are averaging 200 measurements so that the Number is 200. The Average is given to be 64. So the Sum of the 200 measurements is

$$64 \cdot 200 = 12{,}800.$$

Thus, the Sum of the 500 measurements is $21{,}600 + 12{,}800 = 34{,}400$.

Finally, the average of the 500 measurements is $\frac{34{,}400}{500} = 68.8$.

So Quantity A is greater than Quantity B, choice A.

Of 40 numbers in a list, 10 numbers are between 15 and 20 and the remaining 30 numbers are between 50 and 60.

59. Quantity A: The arithmetic mean of the 40 numbers
    Quantity B: 55

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Solution by working with sums:** The greatest possible value of the sum of the 40 numbers is $20 \cdot 10 + 60 \cdot 30 = 200 + 1800 = 2000$. It follows that the greatest possible arithmetic mean is $\frac{2000}{40} = 50$.

So Quantity B is greater than Quantity A, choice B.

Notes: (1) It is just as natural to try to first compute the least possible value of the sum of the 40 numbers. In this case we get

$$15 \cdot 10 + 50 \cdot 30 = 150 + 1500 = 1650,$$

from which it follows that the least possible average is $\frac{1650}{40} = 41.25$.

This narrows the possible choices down to B or D. We still need to compute the greatest possible average to be certain that the answer is B.

(2) It is not entirely clear if the average can actually take on the extreme values 50 and 41.75. This is due to the ambiguity of the word "between."

If we needed to know for certain if these values should be included, then one of the words "inclusive" or "exclusive" would have appeared.

For example, "10 numbers are between 15 and 20, inclusive" would indicate that the numbers 15 and 20 should be included as possibilities.

In this question neither of these words are needed because the answer to the question is the same no matter which we assume.

Math Exam			Chemistry Exam	
Grade	Frequency		Grade	Frequency
100	7		100	1
95	5		95	4
90	5		90	26
85	4		85	2
80	6		80	1
75	8		75	1

60. Quantity A: The standard deviation of grades on the math exam.
    Quantity B: The standard deviation of grades on the chemistry exam.

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

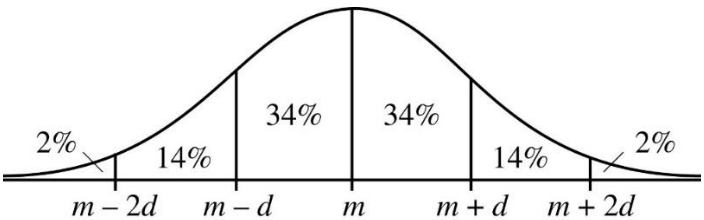
* The scores on the math exam are more "spread out" than the scores on the chemistry exam. It follows that the standard deviation of grades on the math exam is greater, choice A.

Notes: (1) **Standard deviation** measures how far the data values are from the mean. If the data values are close to the mean, the standard deviation is small. If the data is far from the mean, the standard deviation is large.

For example, if all the data values were the same, then the mean would be that common value and the standard deviation would be 0.

(2) The mean of the chemistry grades is approximately 90. Notice that most of the data are near this value (between 85 and 95). So the standard deviation is small.

(3) The mean of the math grades is 87, but the grades are spread out with many values at the extremes 75 and 100. So the standard deviation is larger.



61. The figure above shows a normal distribution with mean m and standard deviation d, including approximate percents of the distribution in each of the six regions shown. For a population of 4000 students in a university, the heights of these students are approximately normally distributed with a mean of 67 inches and a standard deviation of 4 inches. How many of the students had heights between 71 and 75 inches?

A. 80
B. 560
C. 1120
D. 1360
E. 1920

* We are given that $m = 67$ and $d = 4$. It follows that $m + d = 71$ and $m + 2d = 75$. So 14% of the students had heights between 71 and 75. Now, 14% of 4000 is $0.14 \cdot 4000 = 560$, choice B.

62. In September, Daniela was able to type 30 words per minute. In October she was able to type 42 words per minute. By what percent did Daniela's speed increase from September to October?

A. 12%
B. 18%
C. 30%
D. 40%
E. 70%

*** Solution using the percent change formula:** Recall the percent change formula:

$$Percent\ Change = \frac{Change}{Original} \times 100$$

In this question the **original** value is 30. The new value is 42, so that the **change** is $42 - 30 = 12$. Using the percent change formula, we get that the percent increase is $\frac{12}{30} \cdot 100 = 40\%$, choice D.

Warning: Do not accidently use the new value for "change" in the formula. The **change** is the positive difference between the original and new values.

$$15, 17, 3, 19, 2, 5, 22, 36, b$$

63. If b is the median of the 9 numbers listed above, which of the following could be the value of b ? Indicate all such values.

A. 14
B. 15
C. 16
D. 17
E. 18
F. 19
G. 20

* Let's list the numbers in increasing order (without b):

$$2, 3, 5, 15, 17, 19, 22, 36$$

Note that if b is the median (middle number), then $15 \leq b \leq 17$. So the answers are B, C, and D.

Note: Once we have the numbers listed in increasing order, we can strike off one number from each side until we get down to the middle 1 or 2 numbers:

$$\cancel{2}, \cancel{3}, \cancel{5}, 15, 17, \cancel{19}, \cancel{22}, \cancel{36}$$

At this point we see that if b is the median of the list, then $15 \leq b \leq 17$.

64. Three light bulbs are placed into three different lamps. How many different arrangements are possible for three light bulbs of different colors – one white, one black, and one yellow?

Solution by listing: We list all the possibilities, abbreviating each color by using the first letter.

wby wyb bwy byw ywb ybw

We can easily see that there are **6** arrangements.

Remark: When you actually write out this list you should use abbreviations such as “w” for white, “b” for black, and “y” for yellow. This will save some time.

Solution using the counting principle: There are 3 possible lamps to place the white bulb in. After placing the white bulb, there are 2 lamps to place the black bulb in. Finally, there is 1 lamp left to place the yellow bulb in. By the counting principle we get (3)(2)(1) = **6** arrangements.

*** Solution using permutations:** We can count the arrangements without actually making a list. There are 3 light bulbs, and we are arranging all 3 of them. So there are ${}_3P_3 = 3! = 1 \cdot 2 \cdot 3 = \mathbf{6}$ arrangements.

Permutations: ${}_3P_3$ means the number of **permutations** of 3 things taken 3 at a time. In a permutation order matters (as opposed to the **combination** ${}_3C_3$ where the order does not matter).

$${}_3P_3 = \frac{3!}{0!} = \frac{1 \cdot 2 \cdot 3}{1} = 6$$

In general, if n is an integer, then $n! = 1 \cdot 2 \cdot 3 \cdots n$

If n and k are integers, then ${}_nP_r = \frac{n!}{(n-r)!}$

0! = 1 by definition.

LEVEL 3: ARITHMETIC

$$-x = |-3 + |-1||$$

65. Quantity A: x
    Quantity B: 2

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* $|-1| = 1$. So $\big|-3 + |-1|\big| = |-3 + 1| = |-2| = 2$. So we are given that $-x = 2$, and thus, $x = -2$. So Quantity B is greater than Quantity A, choice B.

Note: $|a|$ means the **absolute value** of a. It takes whatever number is between the two lines and makes it nonnegative. Here are a few examples: $|3| = 3$, $|-5| = 5$, $|0| = 0$.

m is a negative integer and n is a nonnegative integer.

66. Quantity A: $m - n$
    Quantity B: $n - m$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* Since n is nonnegative, $-n$ is negative. It follows that $m - n$ is negative.

Similarly, since m is negative, $-m$ is positive. It follows that $n - m$ is positive.

Since $m - n$ is negative and $n - m$ is positive, $m - n < n - m$. So Quantity B is greater than Quantity A, choice B.

Notes: (1) It might help to plug in specific values for m and n in order to understand this solution. For example, let's let $m = -2$ and $n = 3$. Then $m - n = -2 - 3 = -5$ and $n - m = 3 - (-2) = 3 + 2 = 5$.

Observe that in this example, $n - m > m - n$. This observation narrows down the answer to choices B and D.

(2) In general, $n - m$ and $m - n$ have the same absolute value, but different signs. In the example given in note (1), both had absolute value 5, but $m - n$ was negative and $n - m$ was positive.

Using mathematical notation, we have

$$|n - m| = |m - n| \text{ and } n - m = -(m - n).$$

(3) Recall from question 8 that the **integers** consist of the numbers from the set $\{\dots - 3, -2, -1, 0, 1, 2, 3 \dots\}$, and the **positive integers** consist of the numbers from the set $\{1, 2, 3, 4, \dots\}$.

The **negative integers** consist of the numbers from the set

$$\{\dots, -4, -3, -2, -1\}$$

The **nonnegative integers** consist of the numbers from the set

$$\{0, 1, 2, 3, 4, \dots\}$$

Notice that the only difference between the set of positive integers and the set of nonnegative integers is that 0 is nonnegative, but not positive.

k is an even integer and a multiple of 5.

67. Quantity A: The remainder when k is divided by 20
Quantity B: 10

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If we let $k = 30$, then when k is divided by 20, the remainder is 10. So Quantities A and B are equal.

If we let $k = 20$, then when k is divided by 20, the remainder is 0. So Quantities A and B are not equal.

So the answer is D.

Notes: (1) Saying that an integer is even is the same as saying that it is a multiple of 2.

(2) Since 2 and 5 are prime, an integer that is a multiple of 2 and 5 is just a multiple of 10.

(3) Here is the set of multiples of 10:

$$\{\dots -30, -20, -10, 0, 10, 20, 30, \dots\}$$

(4) There are two possible remainders when a multiple of 10 is divided by 20. The remainder can either be 0 or 10.

Important: To find a remainder you must perform division **by hand** or use the algorithm described below. Simply dividing the two numbers in the calculator does **not** give you a remainder!

Calculator Algorithm for computing a remainder: Although performing division in the calculator never produces a remainder, there is a simple algorithm you can perform which mimics long division. Let's find the remainder when 30 is divided by 20 using this algorithm.

Step 1: Perform the division in the calculator: $30/20 = 1.5$
Step 2: Multiply the integer part of this answer by the divisor: $1 \cdot 20 = 20$
Step 3: Subtract this result from the dividend to get the remainder:

$$30 - 20 = \mathbf{10}$$

The original price of a computer was 40 percent less than the computer's \$1600 suggested retail price. The price at which the computer was sold was 30 percent less than the original price.

68. Quantity A: The price at which the computer was sold
    Quantity B: 42% of the computer's suggested retail price

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* The original price of the computer was $0.6 \cdot 1600 = \$960$. So the price at which the computer was sold was $0.7 \cdot 960 = 672$.

42% of the computer's suggested retail price is $0.42 \cdot 1600 = 672$.

So the two Quantities are equal, choice C.

Note: We can decrease 1600 by 40% in two ways.

Method 1: Take 40% of 1600 and then subtract this number from 1600.

$$1600 - 0.4 \cdot 1600 = 1600 - 640 = 960$$

Method 2: Take 60% of 1600 (as was done in the solution above).

$$0.6 \cdot 1600 = 960$$

Decreasing 960 by 30% can also be done by either of these two methods.

69. What is the least integer k such that $\frac{1}{3^k}$ is less than 0.0005 ?

A. 6
B. 7
C. 666
D. 667
E. There is no such least integer

Solution by starting with choice A: Since the word "least" is in the problem, we start with the smallest answer choice. We guess that $k = 6$ and compute $3^k = 3^6 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729$.

So $\frac{1}{3^k} = \frac{1}{729} \approx 0.0013717$.

This number is not less than 0.0005. So we can eliminate choice A.

Let's try B next and guess that $k = 7$. Then $3^k = 3^7 = 729 \cdot 3 = 2187$.

So $\frac{1}{3^k} = \frac{1}{2187} \approx 0.0004572$.

This number is less than 0.0005. So the answer is choice B.

*** Algebraic solution:** We solve the following inequality:

$$\frac{1}{3^k} < 0.0005$$
$$3^k > \frac{1}{0.0005} = 2000$$

We now just multiply 3 by itself until we get a number greater than 2000:

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729 \text{ (so 6 is too small).}$$

Since $729 \cdot 3 = 2187$, the answer is 7, choice B.

70. The ratio of the number of elephants to the number of zebras in a zoo is 3 to 5. If there are 45 elephants in the zoo, how many zebras are in the zoo?

A. 15
B. 60
C. 75
D. 120
E. The answer cannot be determined from the given information

Solution by setting up a ratio:

elephants	3	45
zebras	5	z

Now draw in the division symbols and equal sign, cross multiply and divide the corresponding ratio to find the unknown quantity z.

$$\frac{3}{5} = \frac{45}{z}$$
$$3z = 5 \cdot 45$$
$$3z = 225$$
$$z = \frac{225}{3} = 75$$

This is choice C.

*** Quick solution:** $\frac{45}{3} \cdot 5 = 15 \cdot 5 = 75$, choice C.

Solution by starting with choice C: If we guess that there are 75 zebras at the zoo, then the ratio of the number of elephants to the number of zebras is $\frac{45}{75} = \frac{3}{5}$. This is correct, and so the answer is choice C.

71. Steve entered a number in his calculator and erroneously divided the number by 14,236 instead of 14.236. Which of the following is a single operation that Steve could perform on his calculator to correct the error? Indicate <u>all</u> such operations.

A. Multiply the incorrect quotient by 1000
B. Divide the incorrect quotient by 1000
C. Multiply the incorrect quotient by 10
D. Divide the incorrect quotient by 10
E. Multiply the incorrect quotient by 0.01
F. Divide the incorrect quotient by 0.01
G. Multiply the incorrect quotient by 0.001
H. Divide the incorrect quotient by 0.001

*** Solution by picking a number:** Let's choose a number to divide by, say 100. Let's do the two computations:

Wrong computation: $\frac{100}{14{,}236} \approx 0.0070244$

Correct computation: $\frac{100}{14.236} \approx 7.0244451$

Now let's try all the answer choices:

A. $0.0070244 \cdot 1000 \approx 7.0244$
B. $0.0070244 \div 1000 \approx 0.000007$
C. $0.0070244 \cdot 10 \approx 0.070244$
D. $0.0070244 \div 10 \approx 0.0007024$
E. $0.0070244 \cdot 0.01 \approx 0.0000702$
F. $0.0070244 \div 0.01 \approx 0.70244$
G. $0.0070244 \cdot 0.001 \approx 0.000007$
H. $0.0070244 \div 0.001 \approx 7.0244$

The correct answers are A and H.

Notes: (1) Choices A and H didn't come out exactly the same as when we divided 100 by 14.236 due to the limitations of the GRE calculator.

(2) It's not necessary to do every single computation above if it is clear to you that the wrong answer will be produced. For example, once we know that multiplying by 1000 produces the correct answer, we can eliminate all other choices involving multiplication.

(3) To get from 0.0070244 to 7.0244451 we need to move the decimal point 3 places to the right. This is equivalent to multiplying by 1000 (notice that 1000 has 3 zeros).

Also, multiplying by 1000 is the same as dividing by $\frac{1}{1000} = 0.001$.

72. If a and b are the tens digit and the units digit, respectively, of the product $325{,}189 \times 80{,}577$ what is the value of $b - a$?

* Let's begin simulating the multiplication:

$$\begin{array}{r} {\scriptstyle 6} \\ 325{,}189 \\ \underline{80{,}577} \\ 23 \\ 30 \end{array}$$

Notice that in each row I stopped once I got to the tens digit. That's as far as we need to go to answer the question. I also stopped at the second row because the other rows do not contribute to the units or tens digit.

Now we see that the units digit of the product is $b = 3$ and the tens digit of the product is $a = 2 + 3 = 5$.

So the value of $b - a$ is $3 - 5 = \mathbf{-2}$.

LEVEL 3: ALGEBRA

$$a \neq 0$$

73. Quantity A: $|a| + | - 5|$
    Quantity B: $|a - 5|$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If we let $a = 0$, then we have

$$|a| + |-5| = 0 + 5 = 5 \text{ and } |a - 5| = |0 - 5| = |-5| = 5$$

In this case the two Quantities are equal.

If we let $a = 1$, then we have

$$|a| + |-5| = 1 + 5 = 6 \text{ and } |a - 5| = |1 - 5| = |-4| = 4$$

In this case the two Quantities are <u>not</u> equal.

So the answer is choice D.

$$\frac{a}{b} < 0 < -a, \quad b \neq 0$$

74. Quantity A: a
    Quantity B: 0

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Algebraic solution:** Since $0 < -a$, $-a$ is positive. So a must be negative. In other words, $a < 0$. So Quantity B is greater than Quantity A, choice B.

Notes: (1) We did not need to use any of the information about b to solve this problem.

(2) We can choose values for a and b to come to the same conclusion as in the algebraic solution. If we try to choose a to be nonnegative we should quickly see that $0 < -a$ would be violated. So we will be forced to choose a negative value for a.

For example, we can choose $a = -6$ and $b = 3$. We have $-\frac{6}{3} < 0 < 6$, or equivalently $-2 < 0 < 6$ which is true. Notice that in this case $a < 0$ so that Quantity B is greater than Quantity A. This narrows down our answer to choice B or D.

By going through the process of trying to choose a value for a, it should become clear that a must be negative. So the answer is B.

$$dt \neq 0$$

75. Quantity A: The rate required, in miles per hour, to travel d miles in t hours.

    Quantity B: The rate required, in miles per hour, to travel $\frac{d}{3}$ miles in $3t$ hours.

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution using a formula:** We use the formula

$$\boldsymbol{distance = rate \cdot time} \qquad \text{or} \qquad \boldsymbol{d = r \cdot t}$$

For Quantity A we simply solve the above equation for r to get

$$r = \frac{d}{t}$$

For Quantity B we replace d by $\frac{d}{3}$ and t by $3t$ to get

$$\frac{d}{3} = r(3t)$$

or equivalently,

$$r = \frac{d}{3} \div 3t = \frac{d}{3} \cdot \frac{1}{3t} = \frac{d}{9t} = \frac{1}{9}\left(\frac{d}{t}\right)$$

So we see that Quantity A is greater than Quantity B, choice A.

Note: We can plug in values for d and t to narrow down our choices. For example, let's let $d = 18$ and $t = 2$. Then Quantity A is 9 miles per hour, and Quantity B is 1 mile per hour. So the answer must be choice A or D.

$$g(x) = |x^2 - 1| + |x - 5|$$

76. Quantity A: $g(3)$
    Quantity B: 6

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* $g(3) = |3^2 - 1| + |3 - 5| = |9 - 1| + |-2| = |8| + 2 = 8 + 2 = 10$.

So Quantity A is greater than Quantity B, choice A.

Note: See problem 65 for more information about absolute value.

77. The function k is defined by $k(x) = 2x^2 + bx - 3$, where b is a constant. In the xy-plane, the graph of $y = k(x)$ crosses the x-axis where $x = 3$. What is the value of b ?

    A. 5
    B. 3
    C. 0
    D. -3
    E. -5

* A graph crosses the x-axis at a point where $y = 0$. Thus, the point $(3, 0)$ is on the graph of $y = k(x)$. Equivalently, $k(3) = 0$. So

$$0 = 2 \cdot 3^2 + b(3) - 3$$
$$0 = 2 \cdot 9 + 3b - 3$$
$$0 = 18 + 3b - 3$$
$$0 = 15 + 3b$$
$$3b = -15$$
$$b = -\frac{15}{3} = -5$$

This is choice E.

Note: If f is a function, then

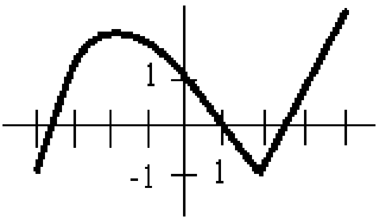
$f(a) = b$ is equivalent to "the point (a, b) lies on the graph of f."

In this problem, the point $(3,0)$ is on the graph of k, and so $k(3) = 0$.

78. If $b = \sqrt[3]{a}$ and $\sqrt{b} = 4$, what is the value of a ?

A. 4096
B. 256
C. 16
D. 4
E. $\sqrt[3]{16}$

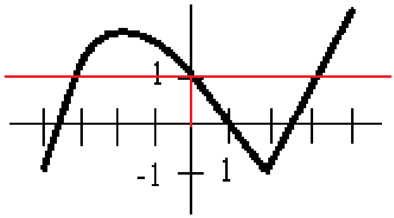
* **Algebraic solution:** We square each side of the equation $\sqrt{b} = 4$ to get $b = 4^2 = 16$. Substituting into the first equation we have $16 = \sqrt[3]{a}$. We now cube each side of this last equation to get $4096 = a$. So the answer is choice A.



79. The figure above shows the graph of the function f . Which of the following are less than $f(0)$? Indicate all such values.

A. $f(-4)$
B. $f(-3)$
C. $f(-2)$
D. $f(-1)$
E. $f(1)$
F. $f(2)$
G. $f(3)$
H. $f(4)$

* Let's draw a horizontal line through the point $(0, f(0))$. To do this start on the x-axis at 0 and go straight up until you hit the curve. This height is $f(0)$. Now draw a horizontal line through this point.



Now, notice that the graph is below this line for the following values of x: $-4, 1, 2,$ and 3. So $f(-4)$, $f(1)$, $f(2)$, and $f(3)$ are all less than $f(0)$. Therefore, the answers are choices A, E, F, and G.

80. For all integers n, the function g is defined as follows

$$g(n) = \begin{cases} n-2 & \text{if} \quad n \text{ is odd} \\ n^2+1 & \text{if} \quad n \text{ is even} \end{cases}$$

What is the value of $g(1) - g(2)$?

* $g(1) = 1 - 2 = -1$ and $g(2) = 2^2 + 1 = 4 + 1 = 5$. So

$$g(1) - g(2) = -1 - 5 = \mathbf{-6}.$$

Notes: (1) The function g in this problem is an example of a **piecewise defined function**. There are two "pieces" here. When evaluating g at an odd integer, we use the first piece, and when evaluating g at an even integer we use the second piece.

(2) Since 1 is odd, we use the first piece to evaluate $g(1)$.

(3) Since 2 is even, we use the second piece to evaluate $g(2)$.

LEVEL 3: GEOMETRY

A small boulder is located on the boundary of a square garden that measures 5 feet on each side. Six bushes are located inside the square garden.

81. Quantity A: The sum of the distances from the boulder to each of the bushes
Quantity B: 45 feet

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* A diagonal of the square has length $5\sqrt{2}$. It follows that the distance from the boulder to one of the bushes is less than $5\sqrt{2}$. Since there are six bushes inside the garden, the sum of the distances from the boulder to each of the bushes is less than $6 \cdot 5\sqrt{2} = 30\sqrt{2}$.

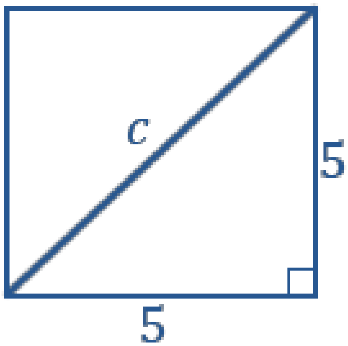
Now $30\sqrt{2} < 30 \cdot 1.5 = 45$. So Quantity B is greater than Quantity A, choice B.

Notes: (1) $(1.5)^2 = (1.5)(1.5) = 2.25 > 2$. It follows that $1.5 > \sqrt{2}$.

So $\sqrt{2} < 1.5$, and therefore $30\sqrt{2} < 30 \cdot 1.5 = 45$.

(2) Instead of using the reasoning given in Note (1), we can simply use the calculator to approximate $30\sqrt{2}$. We type 2, followed by the square root button, and then multiply the result by 30. The display will show 42.426407. Since this is less than 45, the answer is B.

(3) To see that a diagonal of the square has length $5\sqrt{2}$, take a look at the following picture:



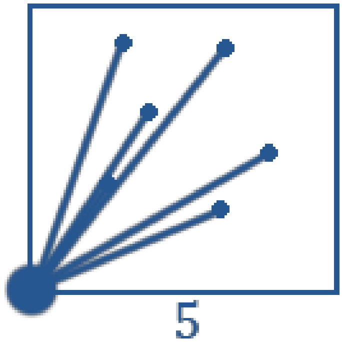
Method 1: We use the Pythagorean Theorem to find c.

$$c^2 = 5^2 + 5^2 = 25 + 25 = 50$$

So $c = \sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$.

Method 2: Observe that the triangle is an isosceles right triangle. This is the same as a 45, 45, 90 triangle. In such a triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of either leg. See Note (4) at the end of problem 20 for details.

(4) To see that the distance from the boulder to each bush is less than the length of a diagonal of the square, take a look at the following picture:



The diagonal of the square is the longest distance between any two points on or inside the square.

In the xy-plane, the points $(a, 1)$ and $(2, b)$ are on the line whose equation is $y = -\frac{1}{2}x + 3$

82. Quantity A: a
    Quantity B: b

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by plugging in the given points:** We substitute each point into the given equation. Let's start with $(a, 1)$.

$$1 = -\frac{1}{2}a + 3$$
$$-2 = a - 6$$
$$4 = a$$

Now let's plug in the point $(2, b)$.

$$b = -\frac{1}{2}(2) + 3$$
$$b = -1 + 3$$
$$b = 2$$

So a is greater than b, choice A.

Note: (1) To get from the first equation to the second equation above (first set of equations), we multiplied each side of the equation by -2. On the left we get -2, and on the right we get

$$-2\left(-\frac{1}{2}a+3\right)=-2\left(-\frac{1}{2}a\right)-2(3)=a-6.$$

(2) To get from the second equation to the third equation above (first set of equations), we added 6 to each side of the equation. On the left we get $-2+6=4$, and on the right we get $(a-6)+6=a$.

ΔABC is isosceles and $\angle ABC = 55°$

83. Quantity A: The sum of the measures of the two angles of ΔABC that have equal measure

    Quantity B: $115°$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* An isosceles triangle has two angles with equal measure. There are two possibilities for Quantity A.

Case 1: The two congruent angles measure $55°$ each. In this case Quantity A is equal to $55+55=110°$, and Quantity B is larger than Quantity A.

Case 2: The sum of the two congruent angles is $180-55=125°$. In this case Quantity A is equal to $125°$, and so Quantity A is greater than Quantity B.

So the answer is D.

Notes: In the first case, the three angles of the triangle measure $55°$, $55°$, and $70°$.

(2) In the second case, the three angles of the triangle measure $55°$, $62.5°$, and $62.5°$.

In the xy-plane, the point $(3,1)$ is on line m, and the point $(-1,-3)$ is on line n. Each of the lines has a negative slope.

84. Quantity A: The slope of line m
    Quantity B: The slope of line n

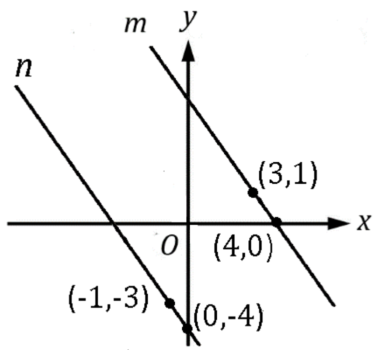
    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* Two points determine a line. Since just one point is given for each line, we can choose any slope (and in particular, any negative slope) we wish for each line. The answer is therefore D.

Note: As an example, suppose that line m passes through the points $(3,1)$ and $(4,0)$. The slope of line m is then $\frac{0-1}{4-3} = -1$.

If we let line n pass through the points $(-1,-3)$ and $(0,-4)$, then the slope of line n is $\frac{-4-(-3)}{0-(-1)} = \frac{-4+3}{1} = -1$. So the two Quantities are the same.

Here is a picture of these two lines on the same set of axes. Notice that they are parallel because they have the same slope.



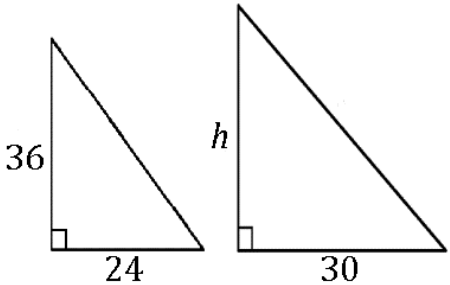
However, if instead we let line n pass through the points $(-1,-3)$ and $(0,-5)$, then the slope of line n is $\frac{-5-(-3)}{0-(-1)} = \frac{-5+3}{1} = -2$. So, in this case, the two Quantities are different.

See if you can draw a picture of line m (through points $(3,1)$ and $(4,0)$) and the last line (through points $(-1,-3)$ and $(0,-5)$) on the same set of axes.

85. A 36 feet tall tree is casting a shadow 24 feet long. At the same time, a nearby tree is casting a shadow 30 feet long. If the lengths of the shadows are proportional to the heights of the trees, what is the height, in feet, of the taller tree?

   A. 54
   B. 45
   C. 42
   D. 36
   E. 30

*** Solution by drawing a figure:** We draw two triangles.



Since we are given that the lengths of the shadows are proportional to the heights of the trees, we have

$$\frac{30}{24} = \frac{h}{36}$$

$$24h = 30 \cdot 36$$

$$h = \frac{30 \cdot 36}{24} = 45$$

This is choice B.

Note: Since corresponding sides of the triangle are in proportion, the triangles are **similar**.

Definition: Two triangles are **similar** if they have the same angle measures.

86. A triangle has a base of length b, an altitude corresponding to that base of length h, and $h = 3b$. Which of the following expresses the area of the triangle, in terms of b ?

A. $\frac{1}{6}b^2$
B. $\frac{3}{4}b^2$
C. b^2
D. $\frac{3}{2}b^2$
E. $6b^2$

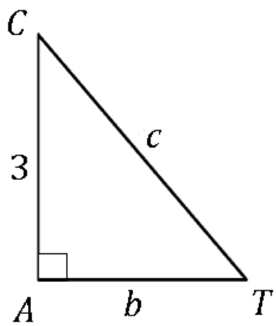
* $A = \frac{1}{2}bh = \frac{1}{2}b(3b) = \frac{3}{2}b^2$, choice D.

Note: The area of a triangle is $\boldsymbol{A = \frac{1}{2}bh}$, where b is the base of the triangle, and h is the height of the triangle.

87. In triangle CAT, the measure of angle A is 90°, the length of side CA is 3, and the length of side AT is b. If the length of hypotenuse CT is between 3 and 6, which of the following could be the value of b ? Indicate <u>all</u> such values.

A. 1
B. 2
C. 3
D. 4
E. 5
F. 6
G. 7

* **Solution using the Pythagorean Theorem:** Let's start by drawing a picture.



By the Pythagorean Theorem, we have $3^2 + b^2 = c^2$, or equivalently, $b^2 = c^2 - 9$.

Let's try the extreme values for c:

$c = 3$: $b^2 = 3^2 - 9 = 9 - 9 = 0$. So $b = 0$

$c = 6$: $b^2 = 6^2 - 9 = 36 - 9 = 27$. So $b = \sqrt{27}$

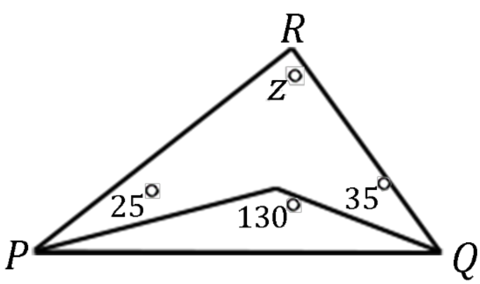
Since $5 < \sqrt{27} < 6$, the answers are A, B, C, D, and E.

Notes: (1) We can use the calculator to approximate $\sqrt{27}$ as 5.1961524.

(2) It is not entirely clear if the extreme values of 0 and $\sqrt{27}$ should be included as solutions. This is due to the ambiguity of the word "between."

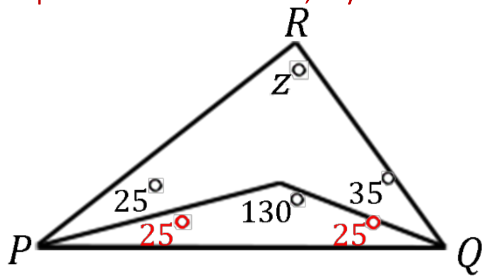
In this question it doesn't matter because neither of those numbers appear as answer choices.

See problem 59 for more about this.



88. In triangle PQR above, what is the value of z ?

Solution by picking numbers: Since every triangle has angle measures which sum to $180°$, we choose values for the angle measures of the small triangle that add up to $180 - 130 = 50°$, say $25°$ and $25°$.



We now see that $z = 180 - 25 - 25 - 35 - 25 = \mathbf{70}$.

Remark: We could have chosen **any** two numbers that add up to 50 for the angles of the small triangle.

*** Direct solution:** The two unlabeled angles in the smaller triangle must have measures that add up to $50°$. So $z = 180 - 25 - 35 - 50 = \mathbf{70}$.

LEVEL 3: DATA ANALYSIS

The average of a, b, c, and 7 is 5.

89. Quantity A: $\frac{a+b+c}{3}$

Quantity B: $\frac{13}{4}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by changing averages to sums:** We change the average to a sum using the formula "**Sum = Average · Number.**"

The Average is given to be 5, and the Number is 4. So the Sum of a, b, c, and 7 is $5 \cdot 4 = 20$. It follows that $a + b + c = 20 - 7 = 13$. So we have $\frac{a+b+c}{3} = \frac{13}{3} > \frac{13}{4}$.

So the answer is A.

Notes: (1) When we increase the denominator of a fraction, the number becomes *smaller*. This is why $\frac{13}{4}$ is smaller than $\frac{13}{3}$.

(2) We can divide in the calculator to get $13/3 \approx 4.33$ and $13/4 = 3.25$. We can then easily see that Quantity A is greater than Quantity B (it is easier to compare decimals than it is to compare fractions).

Solution by picking numbers: We let $a = b = 3$ and $c = 7$. Then the average of a, b, c, and 7 is 5, and $\frac{a+b+c}{3} = \frac{3+3+7}{3} = \frac{13}{3}$. Since $\frac{13}{3} > \frac{13}{4}$, Quantity A is greater than Quantity B, choice A.

Notes: (1) The numbers I chose for a, b, and c were not arbitrary. Since the average of the four numbers a, b, c, and 7 is 5, we need to "balance" the four numbers symmetrically about 5. Since one of the numbers is already 7, the easiest way to do this is to make another of the numbers 7, and the other two numbers 3. This is because 7 and 3 are the same distance from 5.

(2) It may not be entirely clear how we can be certain that the answer is not choice D when we pick numbers. After all, in most problems it would be conceivable that different choices for a, b, and c could lead to a different value for $\frac{a+b+c}{3}$. In this problem that cannot happen however. The reason is that $a + b + c$ is a fixed value (it's 13), even though a, b, and c can vary.

The probability that events A and B will both occur is 0.35.

90. Quantity A: The probability that event B will occur
Quantity B: 0.42

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** We use the **addition principle**:

$$\boldsymbol{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

We have a lot of flexibility here to choose values for $P(A)$ and $P(B)$. For example, we can choose $P(A) = 0.35$ and $P(B) = 0.42$. We then have

$$P(A \text{ or } B) = 0.35 + 0.42 - 0.35 = 0.42.$$

Since the result is a number between 0 and 1, we see that the two Quantities can be equal. This narrows the answer down to C or D.

Now there's no reason that we can't just interchange the values of $P(A)$ and $P(B)$. That is, we can let $P(A) = 0.42$ and $P(B) = 0.35$ to get

$$P(A \text{ or } B) = 0.42 + 0.35 - 0.35 = 0.42.$$

So we see that the two Quantities can be different.

It follows that the answer is D.

Note: A probability must be a number between 0 and 1, inclusive. If we would have wound up with a negative number or a number greater than 1, then the numbers we chose would not have been valid.

91. Quantity A: The standard deviation of a list of 11 different integers, each of which is between 1 and 30, inclusive
Quantity B: The standard deviation of a list of 11 different integers, each of which is between 31 and 60, inclusive

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Quick solution:** The standard deviation measures how far the numbers deviate from the mean, and does not depend upon the values of the numbers themselves. Since each list consists of 11 numbers chosen from a set of 30, the answer is D.

Note: As an example, we can choose the two lists as follows:

A: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

B: $31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41$

Since both lists have each number 1 unit away from the next number, the standard deviations are the same.

As another example, we can choose the two lists as follows:

A: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 30$

B: $31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41$

The only difference between these lists and the previous lists is that the 11 was changed to a 30 in list A. This change increases the standard deviation of list A because 30 is further from the arithmetic mean than 11. So in this case Quantity A is greater than Quantity B.

These two examples together show us that the answer is D.

A list consists of the numbers 3, $\sqrt{11}$, $\sqrt{b}$, and b, where $b > 0$, and the range of the numbers in the list is 12.

92. Quantity A: $\sqrt{b}$
    Quantity B: 4

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

* We are given that $b - 3 = 12$, so that $b = 12 + 3 = 15$. So $\sqrt{b} = \sqrt{15}$. Since $\sqrt{15} < \sqrt{16} = 4$, the answer is B.

Notes: (1) We can also use the calculator to estimate $\sqrt{15}$ as 3.8729833. Since 4 is greater than this number, the answer is B.

(2) The given information does not tell us that the numbers in the list are being given in increasing order. It is therefore conceivable that b is not the greatest number in the list.

(3) In the solution given, we did assume that b is the greatest number. Checking that it is in fact the greatest does require a bit of mathematical sophistication.

(4) Why can't $\sqrt{b}$ be greater than b ? Well in general it can, but this would imply that b and $\sqrt{b}$ are both fractions between 0 and 1. It would then follow that the range would be $\sqrt{11} - b < \sqrt{11} < 12$. So this cannot happen for this problem.

93. A code is created by choosing 2 one digit numbers followed by 2 letters from the 26 letters of the alphabet. Repetition of numbers is allowed, but repetition of letters is not. How many different possible codes can be formed in this way?

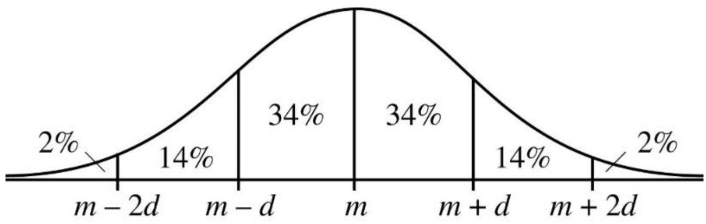
    A. 2,340
    B. 5,850
    C. 6,500
    D. 65,000
    E. 67,600

* We will use the **counting principle** which says that if one event is followed by a second independent event, the number of possibilities is multiplied. So in this example, the number of possible codes is

$$10 \cdot 10 \cdot 26 \cdot 25 = 65{,}000$$

This is choice D.

Remark: In this problem there are actually 4 events. 2 of the events are "choosing a one digit number," the third event is "choosing a letter from the 26 letters of the alphabet," and the fourth event is "choosing a letter from the remaining 25 that have not been chosen already."

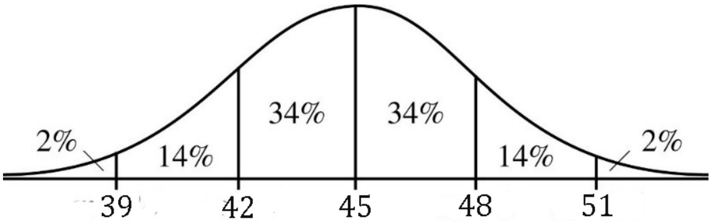


94. The figure above shows a normal distribution with mean m and standard deviation d, including approximate percents of the distribution in each of the six regions shown. Tina's travel times to school last year were approximately normally distributed, with a mean of 45 minutes and a standard deviation of 3 minutes. According to the figure shown, approximately what percent of Tina's travel times to work last year were greater than 42 minutes?

A. 84%
B. 68%
C. 48%
D. 34%
E. 28%

* We are given that $m = 45$ and $d = 3$. It follows that $m - d = 42$. So the desired percentage is $34\% + 50\% = 84\%$, choice A.

Notes: (1) When we replace m with 45 and d with 3, we get the following picture:



(2) According to the last figure drawn, the percent of Tina's travel time to work last year that was between 42 and 45 minutes was 34%.

(3) Since the total percentage is 100, and the graph is symmetrical about the mean, the percent of Tina's travel time to work last year that was greater than 45 minutes was 50%.

Alternatively, we can add $34\% + 14\% + 2\% = 50\%$.

95. The average (arithmetic mean) of the 15 numbers in a list is 16. If the average of 12 of the numbers in the list is between 8 and 14, inclusive, which of the following could be the average of the other 3 numbers? Indicate all possible averages.

A. 16
B. 24
C. 28
D. 72
E. 100
F. 168

*** Solution by changing averages to sums:** We change the averages to sums using the formula "**Sum = Average · Number.**"

First we are averaging 15 numbers so that the Number is 15. The Average is given to be 16. So the Sum of the 15 numbers is

$$16 \cdot 15 = 240.$$

Next we are averaging 12 numbers so that the Number is 12. The Average is given to be between 8 and 14. So the Sum of the 12 numbers is between

$$8 \cdot 12 = 96 \qquad \text{and} \qquad 14 \cdot 12 = 168$$

Thus, the Sum of the other 3 numbers is between $240 - 168 = 72$ and $240 - 96 = 144$.

Finally, the average of the other 3 numbers is between $\frac{72}{3} = 24$ and $\frac{144}{3} = 48$, inclusive. So the answers are B and C.

96. How many different three-digit positive integers are there in which the hundreds digit is greater than 7, the tens digit is less than 5, and the units digit is a multiple of 3 ?

* We will use the **counting principle** to get $2 \cdot 5 \cdot 3 = \mathbf{30}$**.**

Notes: (1) There are 2 digits greater than 7 (8 and 9).

(2) There are 5 digits less than 5 (0, 1, 2, 3, and 4).

(3) There are 3 digits that are multiples of 3 (3, 6, and 9).

LEVEL 4: ARITHMETIC

k is a positive integer.

97. Quantity A: The remainder when k is divided by 7
Quantity B: The remainder when $k + 21$ is divided by 7

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Solution by picking numbers: Let's choose various values for k and compute the two Quantities in each case.

$k = 7$: The remainder when 7 is divided by 7 is 0 (the **quotient** is 1, but there is no remainder). The remainder when $7 + 21 = 28$ is divided by 7 is also 0 (the quotient is 4, but there is no remainder). So the two Quantities are equal.

$k = 8$: The remainder when 8 is divided by 7 is 1 (7 goes into 8 once with 1 left over). The remainder when 29 is divided by 7 is also 1 (7 goes into 29 four times with 1 left over). So, again, the two Quantities are equal.

It seems like the two Quantities will always be equal, and, in fact, the answer is C.

Notes: (1) Recall from problem 67 that to find a remainder you must perform division **by hand** or use the calculator algorithm described in that problem. Simply dividing the two numbers in the calculator does **not** give you a remainder!

(2) Remainders have cyclical behavior. When we divide numbers by 7 the remainders cycle through the numbers 0 through 6, and then repeat forever. Here is a partial table describing this phenomenon. The first row consists of positive integers, and the second row consists of the remainders when we divide these integers by 7.

0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	2	3	4	5	6	0	1	2	3	4	5	6

(3) Be especially careful if you decide to divide numbers less than 7 by 7. For example when we divide 4 by 7, the remainder is 4 (a common error is to put 3). Indeed, 7 goes into 4 zero times, and there are still 4 left.

(4) Due to the cyclical nature of remainders, if we add a multiple of 7 to any integer, the resulting integer will have the same remainder when divided by 7.

For example, 21 is a multiple of 7 (it is $7 \cdot 3$). Therefore k and $k + 21$ always have the same remainder when divided by 7.

*** Quick solution:** Since 21 is a multiple of 7, k and $k + 21$ will always have the same remainder when divided by 7 (see Note (4) in the above solution for more details). So the two Quantities are equal, choice C.

Note for the advanced student: Any integer k can be written in the form $k = 7n + r$, where n and r are integers with $0 \leq r < 7$ (this is an application of something called the **Division Algorithm**). The integer n is called the **quotient**, and r is called the **remainder**. More specifically, we can read the expression "$k = 7n + r$" as "when the integer k is divided by 7, the quotient is n and the remainder is r.

Now, we have

$$k + 21 = (7n + r) + 21 = (7n + 21) + r = 7(n + 3) + r$$

This shows that when $k + 21$ is divided by 7, the quotient is $n + 3$ and the remainder is r. In other words, although k and $k + 21$ have different quotients (n and $n + 3$), their remainders are the same (they are both equal to r).

p is a prime number greater than 2.

98. Quantity A: The number of distinct prime factors of $27p$
    Quantity B: The number of distinct prime factors of $49p$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If we let $p = 3$, then $27p = 27 \cdot 3 = 3^4$, and $49p = 49 \cdot 3 = 3 \cdot 7^2$. So Quantity A is 1 and Quantity B is 2. Therefore, Quantity B is greater than Quantity A.

Now, if we let $p = 7$, then $27p = 27 \cdot 7 = 3^3 \cdot 7$, and $49p = 49 \cdot 7 = 7^3$. So Quantity A is 2 and Quantity B is 1. Therefore, Quantity A is greater than Quantity B.

So the answer is D.

Notes: (1) The only prime factor of 27 is 3 ($27 = 3^3$), and the only prime factor of 49 is 7 ($49 = 7^2$). So it seems natural to try 3 and 7 for p.

(2) If we choose any number for p other than 3 or 7, then Quantities A and B will be the same (they will both be equal to 2). Try setting $p = 5$, for example. I leave the details to the reader.

In August Jennifer had 12 percent more money in the bank than David. In September of the same year the amount of money that Jennifer had in the bank was 12 percent less than the amount she had in the bank in August while the amount of money that David had in the bank remained unchanged.

99. Quantity A: The amount of money that David had in the bank in September

    Quantity B: The amount of money that Jennifer had in the bank in September

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

*** Algebraic solution:** Let D be the amount of money that David had in the bank in August (which is the same as the amount he had in September). The amount that Jennifer had in August was $1.12D$, and therefore the amount she had in September was $0.88 \cdot 1.12D = 0.9856D$. Since this is less than D, Quantity A is greater than Quantity B, choice A.

Notes: (1) To increase D by 12% we multiply D by 1.12 to get $1.12D$.

(2) To decrease $1.12D$ by 12%, we multiply $1.12D$ by $1 - 0.12 = 0.88$ to get $0.88 \cdot 1.12D = 0.9856D$.

(3) If the algebra confuses you, try a specific value for D. Since this is a percent problem, a good choice is $D = 100$ dollars. In this case, David had 100 dollars in the bank in both August and September. Jennifer had 112 dollars in the bank in August, and $0.88 \cdot 112 = 98.56$ dollars in the bank in September. So Quantity A (100) is greater than Quantity B (98.56).

Choice A would be a good guess at this point, but the numbers alone do not rule out choice D as a possibility.

In the set of integers between 1 and 89, A is the set of multiples of 7, B is the set of multiples of 9, and C is the set of multiples of 21.

100. Quantity A: The number of integers that are common to all three sets A, B, and C.

     Quantity B: 4

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Solution by listing: Let's list the members of each set:

$$A = \{7, 14, 21, 28, 35, 42, 49, 56, \mathbf{63}, 70, 77, 84\}$$
$$B = \{9, 18, 27, 36, 45, 54, \mathbf{63}, 72, 81\}$$
$$C = \{21, 42, \mathbf{63}, 84\}$$

Now note that the only number that all three of these sets have in common is 63. So Quantity A is 1, and the answer is B.

Notes: (1) C is the smallest set. It is therefore easiest to see what numbers the 3 sets have in common by going through each number in set C, and then checking if that number is in the other two sets.

For example, we start with 21, and note that 21 is not in set B. So we delete it.

Continuing in this fashion we see that 63 is the only number common to all three sets.

(2) A slightly quicker way of using this method is to list only set C (because 21 is the largest number). Then for each number in set C check if the number is a multiple of both 7 and 9.

*** Direct solution:** The least common multiple of 7, 9, and 21 is 63. So the integers common to all three sets are the multiples of 63 between 1 and 89. 63 is the only such number. So Quantity A is 1, and the answer is B.

Notes: (1) Here is one way to find the least common multiple of 7, 9, and 21 (another method was given in the first solution above).

<u>Step 1:</u> Find the prime factorization of each integer in the set.

$$7 = 7$$
$$9 = 3^2$$
$$21 = 3 \cdot 7$$

<u>Step 2:</u> Choose the highest power of each prime that appears in any of the factorizations.

$$3^2 \text{ and } 7$$

<u>Step 3:</u> Multiply these numbers together to get the least common multiple.

$$3^2 \cdot 7 = 63$$

(2) **Getting the answer quickly:** Starting from 7, write down the prime factors of each number, skipping any that do not contribute to the least common multiple. So we would write 7, then 3^2. We would then think of 21 as $3 \cdot 7$, but we do not write these numbers again because we have already written them. So we have $3^2\ 7$. We then multiply these numbers together to get $3^2 \cdot 7 = 63$.

Definition: The **least common multiple (lcm)** of a set of positive integers is the smallest positive integer that is divisible by each integer in the set.

101. If an integer is divisible by both 27 and 10, then the integer also must be divisible by which of the following?

A. 4
B. 25
C. 36
D. 54
E. 81

*** Solution by picking a number:** Let's choose a number that is divisible by both 27 and 10. An example of such a number is $27 \cdot 10 = 270$. Now we use the calculator to check if 270 is divisible by each of the answer choices.

A. $270/4 = 67.5$
B. $270/25 = 10.8$
C. $270/36 = 7.5$
D. $270/54 = 5$
E. $270/81 \approx 3.333$

Since choices A, B, C, and E did not come out to integers, the answer is D.

Important note: It is very important that you check every answer choice when picking numbers. As we have seen in problem 38, specific numbers can lead to more than one choice coming out correct.

Solution by starting with choice C: $27 = 3^3$ and $10 = 2 \cdot 5$

So we are looking for an answer choice that contains at most the prime factors 2, 3, 3, 3, and 5.

Let's start with choice C and factor $36 = 2^2 \cdot 3^2$. Since the prime factor 2 appears twice (once too much), we can eliminate choice C.

Let's try choice D next and factor $54 = 2 \cdot 3^3$. This works, and so the answer is choice D.

102. If p is the largest prime number less than 25 and q is the smallest prime number greater than 32, then $p + q =$

A. 54
B. 56
C. 58
D. 60
E. 62

* The largest prime number less than 25 is $p = 23$. The smallest prime number greater than 32 is $q = 37$. So $p + q = 23 + 37 = 60$, choice D.

A prime number trick: To test if a number is prime, just check if it is divisible by each prime number up to the square root of the number. For example, let's check if 37 is prime. Well, $\sqrt{37}$ is some number less than 7 (because $7^2 = 49 > 37$). So we need only check if 37 is divisible by 2, 3, and 5.

Since 37 does not end in an even digit, it is not divisible by 2.

Since $3 + 7 = 10$, and 10 is not divisible by 3, it is not divisible by 3.

Since 37 does not end in a 5 or 0, it is not divisible by 5.

Therefore, 37 is prime.

Divisibility tricks: Here are some useful divisibility tricks.

An integer is divisible by 2 precisely when the last digit is 0, 2, 4, 6 or 8.
An integer is divisible by 3 when the sum of its digits is divisible by 3.
An integer is divisible by 4 precisely when the number formed by taking just the last two digits of the integer is divisible by 4.
An integer is divisible by 5 precisely when the last digit is 0 or 5.
An integer is divisible by 6 if it is divisible by 2 and by 3.
An integer is divisible by 9 when the sum of its digits is divisible by 9.
An integer is divisible by 10 if it ends in a 0.

103. An electronics store sells two brands of C batteries at \$3.00 each and \$3.25 each, respectively, and two brands of D batteries at \$4.50 and \$4.75 each, respectively. Which of the following could be the total price of 5 C batteries of one brand and 3 D batteries of one brand? Indicate <u>all</u> such prices.

A. \$28.00
B. \$28.75
C. \$29.25
D. \$29.50
E. \$29.75
F. \$30.00
G. \$30.25
H. \$30.75

*** Solution by listing the possibilities:**

$5 \cdot 3.00 + 3 \cdot 4.50 = 15 + 13.50 = 28.50$
$5 \cdot 3.00 + 3 \cdot 4.75 = 15 + 14.25 = 29.25$
$5 \cdot 3.25 + 3 \cdot 4.50 = 16.25 + 13.50 = 29.75$
$5 \cdot 3.25 + 3 \cdot 4.75 = 16.25 + 14.25 = 30.50$

So the answers are C and E.

104. If k is divided by 9, the remainder is 4. What is the remainder if $3k$ is divided by 9?

Solution by picking a number: Let's choose a positive integer whose remainder is 4 when it is divided by 9. A simple way to find such a k is to add 9 and 4. So let $k = 13$. It follows that $3k = 3 \cdot 13 = 39$. 9 goes into 39 four times with a remainder of **3**.

Note: Recall from problem 67 that to find a remainder you must perform division **by hand** or use the calculator algorithm described in that problem. Simply dividing the two numbers in the calculator does **not** give you a remainder!

*** Quickest solution:** A slightly simpler choice for k is $k = 4$. Indeed, when 4 is divided by 9 we get 0 with 4 left over. Then $3k = 12$, and the remainder when 12 is divided by 9 is **3**.

Note that in general we can get a value for k by starting with any multiple of 9 and adding 4. So $k = 9n + 4$ for some integer n.

Remark: The answer to this problem is independent of our choice for k (assuming that k satisfies the given condition, of course). The method just described does **not** show this. The following solution does.

Complete algebraic solution: The given condition means that we can write k as $k = 9n + 4$ for some integer n. Then

$$3k = 3(9n + 4) = 27n + 12 = 9(3n + 1) + 3 = 9z + 3$$

where z is the integer $3n + 1$. This shows that when $3k$ is divided by 9 the remainder is 3.

LEVEL 4: ALGEBRA

n is an integer for which $\frac{1}{3^{2-n}} > \frac{1}{27}$

105. Quantity A: n
Quantity B: -1

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* **Algebraic solution:** The given inequality is equivalent to $3^{2-n} < 27$. Now $27 = 3^3$, and so we have $3^{2-n} < 3^3$. So we must have $2 - n < 3$. We subtract 2 from each side of this last inequality to get $-n < 1$. Finally, we negate each side to get $n > -1$. So Quantity A is greater than Quantity B, choice A.

Notes: (1) Whenever you take the reciprocal of each side of an inequality, the order is reversed. That's why the first inequality in the solution changed direction from the inequality given in the question.

(2) Similarly, whenever you multiply each side of an inequality by a negative number, the order is reversed. This is why the last inequality in the solution changed direction.

(3) The expression $b^x < b^y$ is equivalent to $x < y$. This is why we were able to replace $3^{2-n} < 3^3$ by $2 - n < 3$.

Solution by plugging in numbers: Let's start by plugging in Quantity B for n. So we let $n = -1$, and we have $\frac{1}{3^{2-n}} = \frac{1}{3^{2+1}} = \frac{1}{3^3} = \frac{1}{27}$. So $n = -1$ *does not* satisfy the given condition. At this point it should be relatively clear that the answer is either A or B. We just need to find a value for n that makes the given inequality true, and then we can use that value for comparison with Quantity B.

Let's try $n = -2$ next. Then $\frac{1}{3^{2-n}} = \frac{1}{3^{2+2}} = \frac{1}{3^4} = \frac{1}{81}$. This is less than $\frac{1}{27}$, and so once again the given condition is *not* satisfied.

So we must have $n > -1$, and the answer is A.

Notes: (1) If it is not clear to you that $\frac{1}{81}$ is less than $\frac{1}{27}$, then you can easily change these fractions to decimals in the calculator. We have $\frac{1}{81} \approx 0.012$ and $\frac{1}{27} \approx 0.037$. It is easier to see that $0.012 < 0.037$.

(2) We may want to try a value for $n > -1$ to be safe. Let's try $n = 0$. Then we have $\frac{1}{3^{2-n}} = \frac{1}{3^{2-0}} = \frac{1}{3^2} = \frac{1}{9}$ which is in fact greater than $\frac{1}{27}$. As in Note (1), you can use your calculator to change these fractions to decimals if you wish to make these numbers easier to compare.

(3) Recall that if the numerator of a positive fraction stays the same, then increasing the denominator makes the value of the fraction smaller.

One of the roots of the equation $x^2 + 5x - c = 0$ is -2, and c is a constant.

106. Quantity A: c
Quantity B: 6

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution using the given root:** Since -2 is a root of the given equation, $(-2)^2 + 5(-2) - c = 0$. So we have $4 - 10 - c = 0$, or equivalently $-6 - c = 0$. Adding c to each side of this last equation gives $-6 = c$. So Quantity B is greater than Quantity A, choice B.

Note: r is a **root** of the equation $f(x) = 0$ if $f(r) = 0$.

In this problem, since -2 is a root of the given equation, we can substitute -2 in for x and solve for c.

Solution using the sum and the product of the roots of a quadratic equation: The sum of the roots of the given quadratic equation is -5. Since one of the roots is -2, the other root must be -3. It follows that the product of the roots is $-c = (-2)(-3) = 6$. So $c = -6$, and therefore Quantity B is greater than Quantity A, choice B.

Notes: (1) If r and s are the roots of the quadratic equation $x^2 + bx + c = 0$, then $b = -(r + s)$ and $c = rs$.

In other words, b is the negative of the sum of the roots, and c is the product of the roots.

(2) In this problem, $b = 5$, and since one of the roots is -2, the other root must be -3.

(3) Be careful of the minus sign in front of c. This means that in this problem, the product of the roots is $-c$ (and not c).

Alternate algebraic solution: Since -2 is a root, $(x + 2)$ is a factor. So we have

$$x^2 + 5x - c = (x + 2)(x + k)$$
$$x^2 + 5x - c = x^2 + kx + 2x + 2k$$
$$x^2 + 5x - c = x^2 + (k + 2)x + 2k$$

Equating corresponding coefficients gives $5 = k + 2$ and $-c = 2k$.

From the first equation, we have $k = 3$. So $-c = 2k = 2 \cdot 3 = 6$, and therefore $c = -6$.

So Quantity B is greater than Quantity A, choice B.

Solution by picking a number: Since -2 is a root, $(x + 2)$ is a factor. So we have

$$x^2 + 5x - c = (x + 2)(x + k)$$

We substitute -2 for x to get $(-2)^2 + 5(-2) - c = 0$.

So we have $4 - 10 - c = 0$, or equivalently $-6 - c = 0$.

Adding c to each side of this last equation we get $-6 = c$.

So Quantity B is greater than Quantity A, choice B.

107. Quantity A: $3x^2 + 5$
 Quantity B: $\sqrt{9x^4 + 30x^2 + 25}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by multiplying:**

$$(3x^2+5)^2=(3x^2+5)(3x^2+5)=9x^4+30x^2+25$$

Since $3x^2+5$ is always positive, it follows that

$$3x^2+5=\sqrt{9x^4+30x^2+25}$$

So both Quantities are equal, and the answer is C.

Notes: (1) Remember that to square an expression means to multiply that expression by itself. So $(3x^2+5)^2=(3x^2+5)(3x^2+5)$.

(2) See problem 41 for two methods of multiplying polynomials.

(3) The equation $a^2=b$ usually has the two solutions $a=\pm\sqrt{b}$ (this is called the **square root property**). If however we know that a cannot be negative, then we need to reject $-\sqrt{b}$.

Since x^2 is always nonnegative, it follows that $3x^2+5$ is always at least 5 (and in particular, it is nonnegative). This is why we get only the positive square root in the above solution.

(4) We can also factor $9x^4+30x^2+25$ to get

$$9x^4+30x^2+25=(3x^2+5)(3x^2+5)=(3x^2+5)^2.$$

It follows that $\sqrt{9x^4+30x^2+25}=\sqrt{(3x^2+5)^2}=3x^2+5$, and so the two Quantities are equal.

(5) Let's plug in any value for x, say $x=0$. We then have $3x^2+5=5$ and $\sqrt{9x^4+30x^2+25}=\sqrt{25}=5$. So the two Quantities are equal. This narrows down the possible answers to C or D.

(6) To increase our chances of being successful using the method given in Note (5) we should try different "types" of values for x. Examples of types might be a positive integer, a positive fraction, a negative integer, and a negative fraction. In this case it might be tedious to try fractions, but we can easily evaluate the two expressions for $x=1$ and $x=-1$. (Do you see that both of these numbers would produce the same result?)

$$a < b < c$$

108. Quantity A: $|a + b + c|$
Quantity B: $|a - b| + |b - c|$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** If we let $a = 0$, $b = 1$, and $c = 2$, then we have $|a + b + c| = |0 + 1 + 2| = |3| = 3$, and we also have $|a - b| + |b - c| = |0 - 1| + |1 - 2| = |-1| + |-1| = 1 + 1 = 2$. So Quantity A is greater than Quantity B.

Now, if $a = -1$, $b = 0$, and $c = 1$, then $|a + b + c| = |-1 + 0 + 1| = 0$, and $|a - b| + |b - c| = |-1 - 0| + |0 - 1| = 1 + 1 = 2$. So Quantity B is greater than Quantity A.

So the answer is D.

109. Samuel obtained a \$14,000 car loan at a simple annual interest rate of r percent. After one year, Samuel made a single payment of \$16,240 to repay the loan, including the interest. What is the value of r ?

A. 8
B. 10.4
C. 14.3
D. 16
E. 18.8

Solution by starting with choice C: We start with choice C and take a guess that $r = 14.3$. Then we have $14{,}000 + 0.143 \cdot 14{,}000 = 16{,}002$. This is too small, and so we can eliminate choices A, B, and C.

We try choice D next and guess that $r = 16$. In this case we have $14{,}000 + 0.16 \cdot 14{,}000 = 16{,}240$. This is correct, and so the answer is D.

Notes: (1) See problem 36 to learn how to convert between decimals and percents. Notice that we converted the percents to decimals before doing any computations. For example, we used 0.143 instead of 14.3.

(2) Notice that to get the final amount of the loan (the initial amount plus the interest), we added the initial amount of $14{,}000$ to our guess for the amount of the interest. We got the latter amount by multiplying our guess for the interest rate (as a decimal) by the initial amount.

For example, when we guessed that $r = 16$, we added the initial amount of $14{,}000$ and our guess of $0.16 \cdot 14{,}000$.

We could have also done this with a single computation as $1.16 \cdot 14{,}000$.

To see why this works simply note that

$$14{,}000 + 0.16 \cdot 14{,}000 = (1 + 0.16) \cdot 14{,}000 = 1.16 \cdot 14{,}000.$$

*** Solution using the simple interest formula:** We use the formula

$$A = P(1 + rt)$$

where P is the initial investment (or **principal**), r is the annual interest rate as a decimal, t is the time in years, and A is the final amount (principal plus interest).

In this problem we are given $P = 14{,}000$, $t = 1$, $A = 16{,}240$, and we are being asked to find r. So we have

$$16{,}240 = 14{,}000(1 + r)$$

There are several ways to proceed at this point.

<u>Method 1</u> (**Start with choice C**): We start with choice C and take a guess that $r = 0.143$. Then we have

$$14{,}000(1 + r) = 14{,}000(1 + 0.143) = 14{,}000(1.143) = 16{,}002.$$

This is too small, so we can eliminate choices A, B, and C.

Let's try choice D next and guess that $r = 0.16$. Then we have

$$14{,}000(1 + r) = 14{,}000(1 + 0.16) = 14{,}000(1.16) = 16{,}240.$$

This is correct, and so the answer is choice D.

<u>Method 2</u> (**Algebraic method 1)** We divide each side of the equation by $14{,}000$ to get $1 + r = \frac{16{,}240}{14{,}000} = 1.16$. So $r = 1.16 - 1 = 0.16$. As a percent, the decimal 0.16 is 16%, choice D.

Method 3 (**Algebraic method 2)** We distribute the $14,000$ on the right to get $16,240 = 14,000 + 14,000r$. We then subtract $14,000$ from each side of this last equation to get $14,000r = 16,240 - 14,000 = 2240$. Finally, we divide each side by $14,000$ to get $r = \frac{2240}{14,000} = 0.16$. As a percent, the decimal $.16$ is 16%, choice D.

Notes: (1) I took the liberty here of using the same variable r as both a decimal and a percent. Technically this is incorrect. The value of r in this problem turns out to be 16, but in the solution when we solve for r, we get 0.16.

If we want the solution to be technically accurate, then we should really replace r by $\frac{r}{100}$ in the equation we want to solve. In other words, the formula we should be using is $A = P\left(1 + \frac{rt}{100}\right)$

Since I felt that this adds an unnecessary layer of confusion, I decided to use this slightly incorrect method.

(2) There are two ways to make the solution given above completely correct (from a technical standpoint). We can change r to $\frac{r}{100}$ as mentioned in Note (2), or we can simply use a different variable name instead of r (such as r^*, r', or another letter completely like k).

110. If $(x + y)(z - w) = 0$, which of the following CANNOT be true?

A. $x = -y$ and $z = w$
B. $x > -y$ and $z = w$
C. $x < -y$ and $z > w$
D. $x = -y$ and $z > w$
E. $x = -y$ and $z < w$

*** Solution by process of elimination:** If $z = w$, then we have

$$(x + y)(z - w) = (x + y)(w - w) = (x + y) \cdot 0 = 0$$

So we can eliminate choices A and B.

If $x = -y$, we have

$$(x + y)(z - w) = (-y + y)(z - w) = 0 \cdot (z - w) = 0$$

So we can eliminate choices D and E. Therefore, the answer is choice C.

Solution by starting with choice C: If $x < -y$ and $z > w$, then $x + y < 0$ and $z - w > 0$. So $(x + y)(z - w)$ is the product of a negative number and a positive number, and is therefore negative. In particular, $(x + y)(z - w) \neq 0$. So the answer is choice C.

111. Which of the following inequalities have at least two solutions between -1 and 1? Indicate all such inequalities.

A. $\frac{3}{7}x < x$
B. $x < x^2$
C. $x - \frac{1}{2} < x + \frac{2}{3}$

*** Algebraic solution:** We solve each inequality individually.

$$\frac{3}{7}x < x$$
$$0 < x - \frac{3}{7}x$$
$$0 < \frac{4}{7}x$$
$$0 < x$$
$$x > 0$$

There are infinitely many real numbers that are greater than 0 and between -1 and 1. Two specific values would be 0.1 and 0.2. So choice A is an answer.

$$x < x^2$$
$$0 < x^2 - x$$
$$0 < x(x - 1)$$

Any negative real number satisfies this last inequality. There are infinitely many negative real numbers between -1 and 1. Two specific values would be $-.1$ and $-.2$. So choice B is an answer.

$$x - \frac{1}{2} < x + \frac{2}{3}$$
$$-\frac{1}{2} < \frac{2}{3}$$

This last inequality is true for all real numbers. So choice C is an answer.

So all three choices are answers: A, B, and C.

Note: This problem can be solved by plugging in numbers and using the calculator right from the beginning. If you're having trouble with the algebra, then this would be the best way for you to do it. I leave the details of this solution to the reader.

112. The total amount of Mark's cell phone bill for the month of April was \$120.75. The bill consisted of a fixed charge of \$82.50 plus a charge of \$0.025 per minute for data usage. For how many <u>hours</u> of data usage was Mark charged for the month of April?

Solution using a linear function: If we let x represent the number of minutes of data usage, and y the total amount of Mark's cell phone bill, then we have $y = 0.025x + 82.50$.

Since we are given that $y = 120.75$, we have $120.75 = 0.025x + 82.50$. We are being asked to find x.

We subtract 82.50 from each side of the equation to get $38.25 = 0.025x$.

We now divide each side of this last equation by 0.025 to get that the number of <u>minutes</u> of data usage was $x = \frac{38.25}{0.025} = 1530$.

We divide this last number by 60 to convert to hours. So the answer is $\frac{1530}{60}$ =**25.5**.

Notes: (1) Recall that the equation of a line in **slope-intercept** form is

$$\boldsymbol{y = mx + b}$$

where m is the slope of the line and $(0, b)$ is the y-intercept of the line.

In this problem, b is the fixed charge of 82.50, and m is the cost per minute for data usage, 0.025.

(2) In general when you have a linear cost function (that is, a cost function of the form $y = mx + b$), b is the fixed cost, and m is called the **marginal cost**. The marginal cost is the additional cost per unit. In this problem a unit is one minute of data usage.

*** Quick solution:** $\frac{120.75-82.50}{0.025} = \frac{38.25}{0.025} = 1530$ and $\frac{1530}{60} = \mathbf{25.5}$.

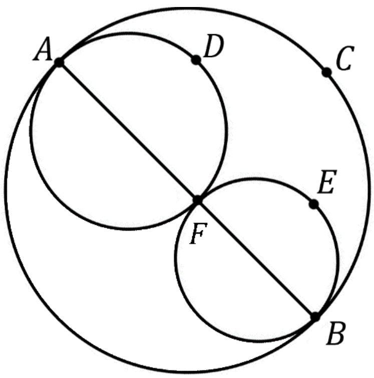
Notes: (1) We subtract the fixed charge of \$82.50 from the total amount of \$120.75 to get \$38.25. This last amount represents the total spent for data usage.

(2) Since it costs $\$0.025$ per minute for data usage, and the amount spent on data usage was $\$38.25$, we get the number of minutes by dividing 38.25 by 0.025.

We can also think of it as multiplication: the amount spent on data usage is equal to the product of the cost per minute and the number of minutes. So we have $28.25 = 0.025x$, where x is the number of minutes. We solve this equation by dividing each side by 0.025.

(3) Once we have that the total number of minutes is 1530, we can convert this number to hours by dividing by 60 because there are 60 minutes in an hour.

LEVEL 4: GEOMETRY



Three circles with their centers on line segment AB are tangent at points A, F, and B, where point F lies on line segment AB.

113. Quantity A: The length of arc ACB
Quantity B: The sum of the lengths of arcs ADF and FEB

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** Let's let $AF = 6$ and $FB = 4$. It follows that $AB = 10$.

Arc ACB is half the circumference of the large circle, and so arc ACB has length $\frac{1}{2} \cdot 10\pi = 5\pi$.

Similarly, arcs ADF and FEB have lengths $\frac{1}{2} \cdot 6\pi = 3\pi$ and $\frac{1}{2} \cdot 4\pi = 2\pi$, respectively. So the sum of the lengths of arcs ADF and FEB is 5π.

For these particular values, the two Quantities are equal. If we pay close attention to the procedure used during this solution, we would notice that it doesn't really matter which numbers were chosen to begin with – the procedure will always lead to the two Quantities being equal. So the answer is C.

Notes: (1) The circumference of a circle is $C = 2\pi r$, where r is the radius of the circle.

Alternatively, we can write $C = \pi d$, where d is the diameter of the circle.

(2) I chose even numbers for the length of each diameter because I knew that I would need to be using half of these numbers. This wasn't essential, but it made the computations easier.

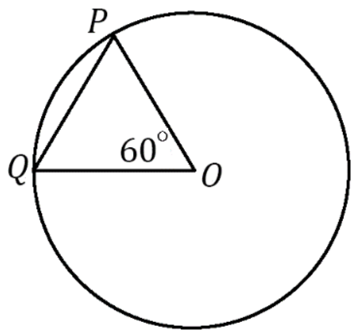
(3) This method of solution narrowed down the answer to choice C or D. C would certainly be the better guess here as it is unlikely that these two computations would yield identical numbers for the values chosen unless the Quantities were always equal. Furthermore, following the procedure carefully should lead us to believe that the Quantities would come out the same regardless of the numbers chosen.

*** Algebraic solution:** If we let $AF = a$ and $FB = b$, then we have that $AB = a + b$.

The length of arc ACB is then $\frac{1}{2}(a + b)\pi$.

The length of arc ADF is $\frac{1}{2}a\pi$, and the length of arc FEB is $\frac{1}{2}b\pi$. So the sum of the lengths of arcs ADF and FEB is $\frac{1}{2}a\pi + \frac{1}{2}b\pi = \frac{1}{2}(a + b)\pi$.

We see that the two Quantities are equal, choice C.



O is the center of the circle and the perimeter of ΔPOQ is 10.

114. Quantity A: The circumference of the circle
Quantity B: 20

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* The triangle is equilateral, and therefore each side of the triangle has length $\frac{10}{3}$. Since OQ is a radius of the circle, the radius of the circle is $\frac{10}{3}$**.** It follows that the circumference of the circle is $\frac{20}{3}\pi$ which is greater than 20. So Quantity A is greater, and the answer is A.

Notes: (1) When a triangle is inside a circle, very often the triangle will be isosceles (it will have two sides of equal length). This happens when two of the sides are radii of the circle (all radii of a circle are equal). The two angles opposite these radii will then have equal measure.

(2) An isosceles triangle with a 60° angle is always equilateral.

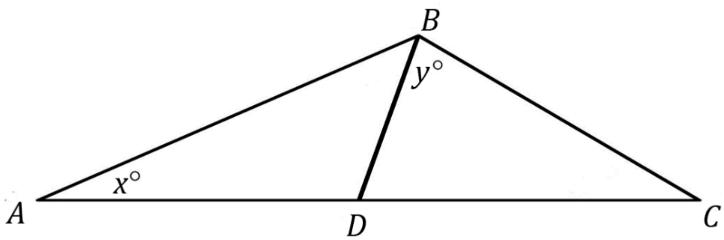
(3) Here is a more detailed argument that shows that the given triangle is equilateral: Since OP and OQ are both radii of the circle, the triangle is isosceles. It follows that angles P and Q have the same measure. Since the angles of a triangle sum to 180°, each of those angles have measure $\frac{180-60}{2} = \frac{120}{2} = 60°$. It follows that the triangle is equilateral.

(4) We are given that the perimeter of the triangle is 10. Since all three sides have equal length, each side has length $\frac{10}{3}$.

(5) The circumference of a circle is $2\pi r$, where r is the radius of the circle. In this problem, $r = \frac{10}{3}$, and so the circumference is $2\pi\left(\frac{10}{3}\right) = \frac{20\pi}{3}$.

(6) $\pi \approx 3.14$. Since $\pi > 3$, we have that $\frac{\pi}{3} > 1$. So $\frac{20\pi}{3} = 20\left(\frac{\pi}{3}\right) > 20(1)$. So $\frac{20\pi}{3} > 20$.

(7) We can use the approximate value of 3.14 for π, and substitute in to the calculator to get $20\left(\frac{\pi}{3}\right) \approx 20\left(\frac{3.14}{3}\right) \approx 20.933333 > 20$.

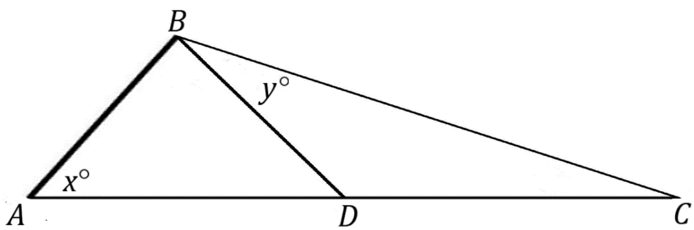


$AD = DC$

115. Quantity A: x
Quantity B: y

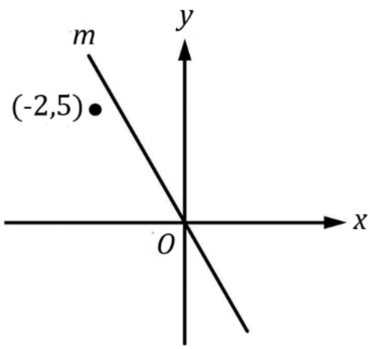
A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by drawing another picture:** We draw another representation of a triangle that satisfies the given condition.



Now observe that in this new figure it looks like $x > y$, whereas in the given figure it looks like $x < y$. So the answer is choice D.

Note: The new triangle we drew needed only to satisfy $AD = DC$.



116. Quantity A: The slope of line m
Quantity B: -2

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* Let's begin by computing the slope of the line passing through the points $(0,0)$ and $(-2,5)$. This slope is $\frac{5-0}{-2-0} = -\frac{5}{2}$. Now note that the slope of line m is smaller than this slope. So we have

$$\text{Slope of } m < -\frac{5}{2} = -2.5 < -2$$

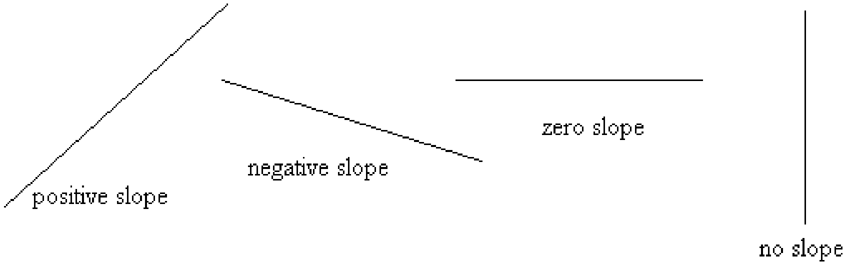
So Quantity B is greater than Quantity A, choice B.

Notes: (1) The **slope** of a line is

$$\text{Slope} = m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

(2) If one of the points on the line is the origin $(0,0)$, then the slope is simply $\frac{y_1}{x_1}$, where (x_1, y_1) is the point that is not the origin.

(3) Lines with positive slope have graphs that go upwards from left to right. Lines with negative slope have graphs that go downwards from left to right. If the slope of a line is zero, it is horizontal. Vertical lines have **no** slope (this is different from zero slope).



(4) If we are comparing two positive slopes, then the line with the greater slope will be the one that is closer to vertical.

If we are comparing two negative slopes, then the line with the smaller slope will be the one that is closer to vertical.

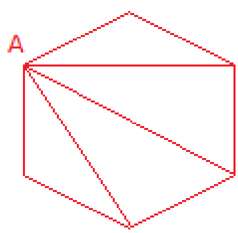
(5) Note that $\frac{5}{2} = 2.5 > 2$. It follows that $-\frac{5}{2} < -2$.

Remember that when we negate each side of an inequality, the inequality symbol reverses.

117. Point A is a vertex of a 6-sided polygon. The polygon has 6 sides of equal length and 6 angles of equal measure. When all possible diagonals are drawn from point A in the polygon, how many triangles are formed?

A. One
B. Two
C. Three
D. Four
E. Six

*** Solution by drawing a picture:**

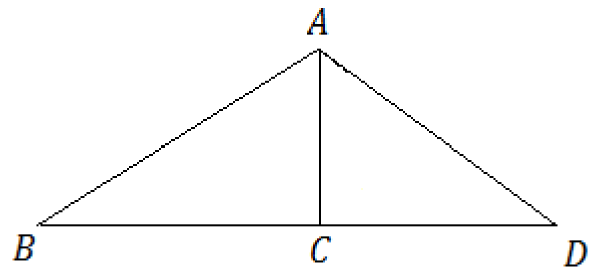


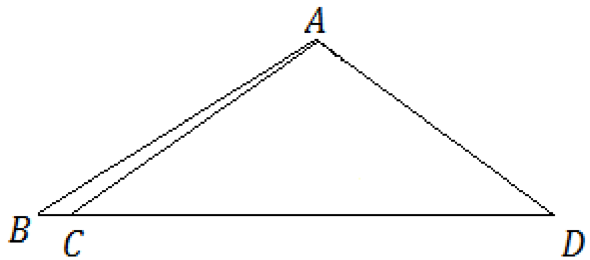
Observe that there are four triangles, choice (D).

118. Points B, C, and D lie on a line in that order, and point A is not on the line. If $AB = AD$, which of the following must be true?

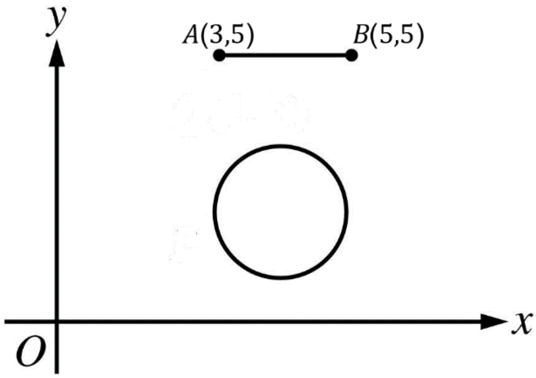
A. $AB > AC$
B. $AB > BC$
C. $AB > BD$
D. $AC > CD$
E. $BC > CD$

*** Solution by drawing pictures:**





* The shortest distance from A to BD is the perpendicular distance as shown in the first figure above. As we move point C to the left or right along AD, the length of AC increases. In this way, we see that AB and AD will both be larger than AC as long as C is between B and D. Therefore, the answer is choice A.



119. The figure shows line segment AB and a circle with radius 1 and center (4,2) in the xy-plane. Which of the following values could be the distance between a point on line segment AB and a point on the circle? Indicate **all** such values.

A. 1.5
B. 2.0
C. 2.5
D. 3.0
E. 3.5
F. 4.0
G. 4.5
H. 5.0
I. 5.5
J. 6.0

* The shortest distance between a point on the segment and a point on the circle will be the distance between the points (4,5) and (4,3). This distance is $5 - 3 = 2$.

The furthest such distance will be the distance between (3,5) and (4,1). This distance is

$$\sqrt{(4-3)^2 + (1-5)^2} = \sqrt{1^2 + (-4)^2} = \sqrt{1+16} = \sqrt{17} < 4.5$$

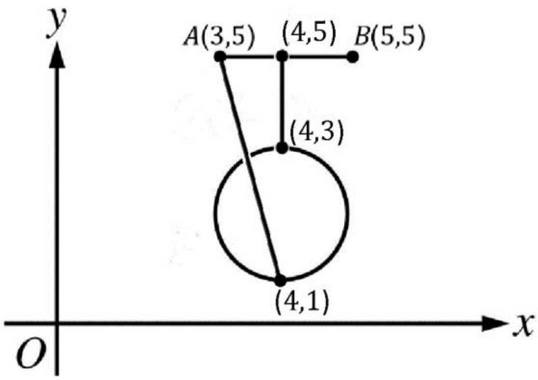
So the answers are B, C, D, E, and F.

Notes: (1) The distance between the two points (x, y) and (z, w) is given by

$d = \sqrt{(z-x)^2 + (w-y)^2}$ or equivalently $d^2 = (z-x)^2 + (w-y)^2$

This formula is called the **distance formula**.

(2) Here is a picture displaying the two extreme distances.



120. If the circumference of circle O is 5 times the circumference of circle O', then the area of circle O is how many times the area of circle O' ?

Solution by picking numbers: Let's let the circumference of circle O be10π. It follows that the circumference of circle O' is 2π.

The radius of circle is O is 5, and the radius of circle O' is 1.

So the area of circle O is 25π, and the area of circle O' is π.

So we see that the area of circle O is $\mathbf{25}$ times the area of circle O'.

Notes: (1) The circumference of a circle is $C = 2\pi r$, and the area of a circle is $A = \pi r^2$.

(2) Since the circumference of circle O is $2\pi r = 10\pi$, it follows that the radius of circle O is $r = \frac{10\pi}{2\pi} = 5$.

Similarly, since the circumference of circle O' is 2π, it follows that the radius of circle O' is $r' = \frac{2\pi}{2\pi} = 1$.

(3) Since the radius of circle O is $r = 5$, it follows that the area of circle O is $A = \pi r^2 = \pi \cdot 5^2 = 25\pi$.

Similarly, since the radius of circle O' is $r' = 1$, it follows that the area of circle O' is $A' = \pi(r')^2 = \pi \cdot 1^2 = \pi$.

Algebraic solution: Let r, C, and A be the radius, circumference and area of circle O, and let r', C', and A' be the radius, circumference and area of circle O'.

We are given that $C = 5C'$, so that $2\pi r = 5(2\pi r')$. It then follows that $r = 5r'$, so $A = \pi r^2 = \pi(5r')^2 = \pi \cdot 25 \cdot (r')^2 = 25\pi(r')^2 = 25A'$.

So we see that the area of circle O is $\mathbf{25}$ times the area of circle O'.

* **Quick solution:** Since circumference is a linear function of the radius ($C = 2\pi r$), multiplying the circumference by 5 multiplies the radius by 5.

Since area is a quadratic function of the radius ($A = \pi r^2$), multiplying the radius by 5 multiplies the area by $5^2 = \mathbf{25}$.

LEVEL 4: DATA ANALYSIS

The average (arithmetic mean) of a, b , and 17 is 11.

121. Quantity A: The average of a and b
Quantity B: 8.1

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* **Solution by changing averages to sums:** We change the given average to a sum using the formula "**Sum = Average · Number.**"

So the Sum of the 3 numbers is $11 \cdot 3 = 33$.

Thus, $a + b + 17 = 33$, and it follows that $a + b = 16$.

So the average of the 2 numbers is $\frac{a+b}{2} = \frac{16}{2} = 8$.

So Quantity B is greater than Quantity A, choice B.

Solution by picking numbers: Let's let $a = 5$ and $b = 11$. We make this choice because 5 and 17 are both 6 units from 11. Then the average of a and b is $\frac{a+b}{2} = \frac{5+11}{2} = \frac{16}{2} = 8$.

So Quantity B is greater than Quantity A, choice B.

A wall is to be painted one color with a stripe of a different color running through the middle. Seven different colors are available.

122. Quantity A: The number of possible color combinations for the wall and the stripe.
Quantity B: 28

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution using the counting principle:** The counting principle says that when you perform events in succession you multiply the number of possibilities. There are 7 ways to choose a color for the wall. Once this color is chosen there are now 6 ways to choose a color for the stripe. Thus, there are $7 \cdot 6 = 42$ possibilities. So Quantity A is greater than Quantity B, choice A.

Solution by making a list: Let's assume the colors are red, blue, green, yellow, purple, orange, and white. We will list all the possibilities in a nice way:

RB, RG, RY, RP, RO, RW
BR, BG, BY, BP, BO, BW
GR, GB, GY, GP, GO, GW
YR, YB, YG, YP, YO, YW
PR, PB, PG, PY, PO, PW
OR, OB, OG, OY, OP, OW
WR, WB, WG, WY, WP, WO

In the above list, we abbreviated each color by using the first letter of its name. The first position is for the wall, and the second for the stripe. We see that we have listed 42 possibilities. So Quantity A is greater than Quantity B, choice A.

Remark: I don't recommend solving the problem this way on the GRE. However, while practicing GRE problems it is a good idea to write out these lists until you have a good understanding of why the counting principle works.

Solution using permutations: There are ${}_7P_2 = 7 \cdot 6 = 42$ ways to choose 2 colors from 7, and place them in a specific order. So Quantity A is greater than Quantity B, choice A.

Important note: Don't let the word "combinations" in the problem itself trick you. This is **not** a combination in the mathematical sense. If you paint the wall red and the stripe blue, then this is a **different** choice from painting the wall blue and the stripe red.

One animal is to be selected from a group of 60 animals. The probability that the selected animal will be a cat is 0.35, and the probability that the selected animal will be a female cat is 0.2.

123. Quantity A: The number of male cats in the group
Quantity B: 10

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* The number of cats in the group is $0.35 \cdot 60 = 21$, and the number of female cats in the group is $0.2 \cdot 60 = 12$. It follows that the number of male cats in the group is $21 - 12 = 9$. So Quantity B is greater than Quantity A.

The average height of the boys in a class is 71 inches and the average height of the girls in the class is 62 inches. The average height for the students in the class is 65 inches.

124. Quantity A: The number of girls in the class
Quantity B: The number of boys in the class

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Solution by changing averages to sums: We change the averages to sums using the formula "**Sum = Average · Number.**"

If we let b be the Number of boys in the class, and we let g be the number of girls in the class, then we get

$$71b + 62g = 65(b + g)$$
$$71b + 62g = 65b + 65g$$
$$6b = 3g$$
$$g = \frac{6}{3}b = 2b$$

So the number of girls in the class is twice the number of boys in the class. Therefore, Quantity A is greater than Quantity B, choice A.

Notes: (1) Since the Average height of the boys is 71 inches, and the Number of boys is b, the Sum of the heights of the boys is $71b$.

(2) Since the Average height of the girls is 62 inches, and the Number of girls is g, the Sum of the heights of the girls is $62g$.

(3) Since the Average height of all the students is 65 inches, and the Number of students is $b + g$ (the total number of students is the number of boys plus the number of girls), the Sum of the heights of all the students is $65(b + g)$.

(4) We use the distributive property to get $65(b + g) = 65b + 65g$.

(5) To get from the second to the third equation in the solution, we subtract $65b$ from each side of the equation, and we also subtract $62g$ from each side of the equation.

(6) Be careful interpreting the equation $g = 2b$. It might help to plug in a number for b. For example, if we let $b = 1$, then $g = 2 \cdot 1 = 2$. So there are more girls than boys.

(7) We can try to pick a number to narrow down the answer choices, but this turns out to be a bit messy in this problem. For example, let's assume that there are 2 boys, both with height 71 inches (so that the average of the 2 boys' heights is 71). Let's assume that all the girls have height 62 inches (so the average of all the girls' heights is 62). We need to figure out how many girls we need so that the overall average is 65 inches. If we let g be the number of girls, then we need $62g + 71 \cdot 2 = 65(g + 2)$. So $62g + 142 = 65g + 130$. We now subtract $62g$ from each side of this last equation to get that $142 = 3g + 130$. Subtracting 130 from this equation gives $12 = 3g$. So $g = \frac{12}{3} = 4$. So there are 2 boys and 4 girls. This narrows down the answer to choice A or D.

*** Quick solution:** Since the average height for all the students is closer to the average height of the girls than the boys, there must be more girls than boys. So the answer is A.

Note: $65 - 62 = 3$ and $71 - 65 = 6$. Since 3 is smaller than 6, the average height for all students is closest to the average height of the girls.

125. How many four-digit positive integers are there such that the four digits are 2, 3, 6, and 9 ?

A. 10
B. 20
C. 24
D. 256
E. 324

*** Solution using the counting principle:** The counting principle says that when you perform events in succession you multiply the number of possibilities. There are 4 ways to choose the leftmost digit $(2, 3, 6, \text{or } 9)$. Once this digit has been chosen there are 3 ways to choose the next digit to the right. Then there are 2 ways to choose the next digit, and finally just 1 way to choose the rightmost digit. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such possible integers, choice C.

Solution by making a list: Let's list all of the possibilities in increasing order:

2369	2396	2639	2693	2936	2963
3269	3296	3629	3692	3926	3962
6239	6293	6329	6392	6923	6932
9236	9263	9326	9362	9623	9632

There are 24 integers in this list, choice C.

Remark: I don't recommend solving the problem this way on the GRE. However, while practicing GRE problems it is a good idea to write out these lists until you have a good understanding of why the counting principle works.

Solution using permutations: There are ${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to arrange 4 numbers, choice C.

126. Of the 1500 students in a high school, 660 are female and $\frac{1}{4}$ of the female and $\frac{1}{8}$ of the male students are freshmen. If one of the students is chosen at random, what is the probability that the student chosen will be a freshman?

A. $\frac{2}{25}$
B. $\frac{4}{25}$
C. $\frac{9}{50}$
D. $\frac{1}{5}$
E. $\frac{1}{4}$

* Since $\frac{1}{4}$ of the female students are freshman, there are $\frac{1}{4} \cdot 660 = 165$ female freshman.

The total number of male students is $1500 - 660 = 840$. Since $\frac{1}{8}$ of the male students are freshman, there are $\frac{1}{8} \cdot 840 = 105$ male freshman.

The total number of freshman is then $165 + 105 = 270$.

Since there are 270 freshman and 1500 students in total, the probability a randomly selected student will be a freshman is $\frac{270}{1500} = \frac{9}{50}$, choice C.

Note: To compute a simple probability where all outcomes are equally likely, divide the number of "successes" by the total number of outcomes.

In this problem, the successes are the 270 freshman, and the total number of outcomes is 1500.

127. 465 pet owners were surveyed to determine whether they owned cats or dogs. 355 said they had at least one cat and 260 said they had at least one dog. Assuming that every couple surveyed had at least one cat or dog, which of the following statements must be true? Indicate **all** such statements.

A. More than half of the cat owners are also dog owners
B. More than half of the dog owners are also cat owners
C. None of the pet owners have both cats and dogs

* We use the formula

$$\text{Total} = C + D - \text{ Both} + \text{Neither}$$

$$\text{Total} = 465,\ C = 355, D = 260, \text{ and Neither} = 0.$$

$$465 = 355 + 260 - \text{Both} + 0 = 615 - \text{Both}$$

So $\text{Both} = 615 - 465 = 150$.

Since $150 < \frac{1}{2} \cdot 355$, choice A is false.

Since $150 > \frac{1}{2} \cdot 260$, choice B is true.

Since we computed that 150 of the pet owners have both cats and dogs, choice C is false.

So the only answer is choice B.

Note: $\frac{1}{2} \cdot 355 = 177.5$ and $\frac{1}{2} \cdot 260 = 130$.

List A: 7, a, b, c
List B: 2, 3, 6, a, b, c

128. If the average (arithmetic mean) of the 4 numbers in list A is 10, what is the average of the 6 numbers in list B? Express your answer as a reduced fraction.

* **Solution by changing averages to sums:** For list A we change the average to a sum using the formula "**Sum = Average · Number.**"

We are given that the Average of the numbers in list A is 10, and the Number is 4. So the Sum of the numbers in list A is $10 \cdot 4 = 40$. So we have $7 + a + b + c = 40$, or $a + b + c = 40 - 7 = 33$.

It follows that the sum of the numbers in list B is

$$2 + 3 + 6 + a + b + c = 11 + a + b + c = 11 + 33 = 44.$$

So the average of the 6 numbers in list B is $44/6 = \mathbf{22/3}$.

Note: This problem can also be solved by picking numbers. Letting $a = 7$ and $b = c = 13$ would be a good choice. See problem 89 for further explanation of this choice for a, b, and c. I leave the details of this solution to the reader.

LEVEL 5: ARITHMETIC

m and n are integers greater than 2.

129. Quantity A: $\frac{m-1}{n-1}$

Quantity B: $\frac{m+2}{n+2}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** Let's let $m = n = 3$. Then we have $\frac{m-1}{n-1} = \frac{3-1}{3-1} = \frac{2}{2} = 1$ and $\frac{m+2}{n+2} = \frac{3+2}{3+2} = \frac{5}{5} = 1$. So Quantities A and B are equal.

Now let's try $m = 3$ and $n = 4$. Then $\frac{m-1}{n-1} = \frac{3-1}{4-1} = \frac{2}{3} \approx 0.667$ and $\frac{m+2}{n+2} = \frac{3+2}{4+2} = \frac{5}{6} \approx 0.833$. So Quantities A and B are not equal.

So the answer is D.

Note: It is easy to see that if $m = n$, then both Quantities are equal to 1. So the first computation in this solution was not really necessary.

Algebraic solution: The following equations are equivalent for $m, n > 2$:

$$\frac{m-1}{n-1} = \frac{m+2}{n+2}$$
$$(m-1)(n+2) = (n-1)(m+2)$$
$$mn + 2m - n - 2 = mn + 2n - m - 2$$
$$2m - n = 2n - m$$
$$3m = 3n$$
$$m = n$$

This shows that Quantities A and B are equal if and only if $m = n$. So the answer is D.

Notes: (1) To get from the first equation to the second equation we cross multiplied.

(2) To get from the second equation to the third equation we can use any method we like for multiplying polynomials such as FOIL, or the standard algorithm that has been used throughout this book (see problem 41).

(3) Observe that each side of the third equation has the terms mn and -2. We can therefore simply delete these terms from each side of the equation. This is how we got the fourth equation.

(4) To get from the fourth equation to the fifth equation we add m to each side of the equation, and we add n to each side of the equation.

(5) To get from the fifth equation to the sixth equation we simply divided each side of the equation by 3.

(6) A similar computation can be used to show that $\frac{m-1}{n-1} < \frac{m+2}{n+2}$ if and only if $m < n$.

Although not needed to solve this problem, this shows that it is possible to choose m and n so that Quantity A is greater, Quantity B is greater, or the Quantities are equal.

L is a list of 200 different numbers that are between 0 and 40. The number a is less than 70 percent of the numbers in L, and the number b is less than 30 percent of the numbers in L.

130. Quantity A: $b - a$
Quantity B: 38

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

*** Solution by picking numbers:** a and b could both be the smallest number in the list. In this case we have $b - a = 0$ and Quantity B is greater than Quantity A.

Now we can have 198 of the numbers between 39 and 40, and the other number equal to 0.1. Then we can choose $a = 0.1$ and $b > 39$. So we have $b - a > 39 - 0.1 = 38.9 > 38$. In this case we have Quantity A greater than Quantity B.

So the answer is D.

Note: The real numbers form a set which is **dense**. This means that between any two real numbers there is another real number.

This density property is what allows us to choose 198 distinct numbers between 39 and 40. For example, we can choose the first number to be 39.5. We can then choose the second number to be 39.8. Maybe we'll choose the third number to be 39.92, and the fourth number to be 39.97. Density of the reals allows us to keep choosing real numbers between the last choice and 40, no matter how close our choices get to 40.

131. If $\frac{st}{u}$ is an integer, which of the following must also be an integer?

   A. stu
   B. $\frac{3s^2t^2}{u^2}$
   C. $\frac{su}{t}$
   D. $\frac{tu}{s}$
   E. $\frac{s}{tu}$

Solution by picking numbers: Let's pick numbers for s, t, and u. Let's try $s = 3$, $t = 4$, and $u = 2$. Then $\frac{st}{u} = \frac{3 \cdot 4}{2} = \frac{12}{2} = 6$, an integer. So the given condition is satisfied. Now let's check the answer choices.

   A. $3 \cdot 4 \cdot 2 = 24$
   B. $3 \cdot 9 \cdot 16/4 = 108$
   C. $3 \cdot 2/4 = 1.5$
   D. $4 \cdot 2/3 \approx 2.67$
   E. $3/(4 \cdot 2) = 0.375$

We can eliminate choices C, D, and E because they are not integers.

Let's try another set of numbers, say $s = 0.5$, $t = 1$, and $u = 0.5$. Then we have $\frac{st}{u} = \frac{0.5 \cdot 1}{0.5} = 1$, an integer. Let's check the answer choices that still remain.

   A. $0.5 \cdot 1 \cdot 0.5 = 0.25$
   B. $3 \cdot 0.25 \cdot 1/0.25 = 3$

So we can eliminate choice A, and the answer is B.

*** Quick solution:** Let $\frac{st}{u}$ be an integer. Then $\frac{3s^2t^2}{u^2} = 3\left(\frac{st}{u}\right)^2$ is an integer because the product of two integers is an integer.

Note: If you are having trouble seeing that the expression in the algebraic solution is an integer, the following might help. Since $\frac{st}{u}$ is an integer, we can write $\frac{st}{u} = k$ for some integer k. It follows that $\frac{3s^2t^2}{u^2} = 3\left(\frac{st}{u}\right)^2 = 3k^2$.

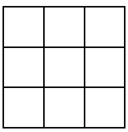
Note that $3k^2$ is an integer because the set of integers is closed for multiplication.

132. In an empty square field, n rows of n trees are planted so that the whole field is filled with trees. If k of these trees lie along the boundary of the field, which of the following is a possible value for k ?

A. 14
B. 49
C. 86
D. 125
E. 276

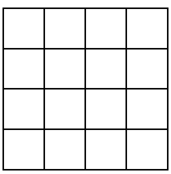
* We will systematically try values for n, and draw a picture of the situation to determine the corresponding value for k.

Here is the picture for $n = 3$.



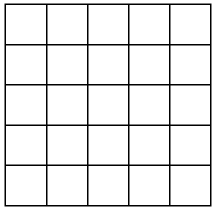
Note that $k = 9 - 1 = 8$.

Here is the picture for $n = 4$.



Note that $k = 16 - 4 = 12$.

Here is the picture for $n = 5$.



Note that $k = 25 - 9 = 16$.

So the pattern appears to be $8, 12, 16, 20, 24, 28, \ldots$

Make sure that you keep drawing pictures until this is clear to you.

So we see that the answer must be divisible by 4.

Beginning with choice C we have $\frac{86}{4} = 21.5$. So choice C is not the answer.

We can eliminate choices B and D because they end in an odd digit.

Trying choice E we have $\frac{276}{4} = 69$. Thus, 276 is divisible by 4, and the answer is choice E.

For the advanced student: Let's prove that for each n, the corresponding k is divisible by 4.

For fixed n, the total number of trees is n^2, and the number of trees **not** on the boundary is $(n-2)^2 = n^2 - 4n + 4$. Thus, the number of trees on the boundary is

$$k = n^2 - (n^2 - 4n + 4) = n^2 - n^2 + 4n - 4 = 4n - 4 = 4(n-1)$$

which is divisible by 4.

133. If a and b are positive integers and k is the greatest common factor of a and b, then k must be the greatest common factor of a and which of the following integers?

A. $a + b$
B. $3 + b$
C. ab
D. $3b$
E. b^2

Solution by picking numbers: Let's try $a = 12$ and $b = 18$. Then $k = 6$. Now lets' check if 6 is the greatest common factor of a and each answer choice.

A. $a + b = 12 + 18 = 30$ $\quad$ $\gcd(12, 30) = 6$
B. $3 + b = 3 + 18 = 21$ $\quad$ $\text{gcf}(12, 21) = 3$
C. $ab = 12 \cdot 18 = 216$ $\quad$ $\text{gcf}(12, 216) = 12$
D. $3b = 3 \cdot 18 = 54$ $\quad$ $\text{gcf}(12, 54) = 6$
E. $b^2 = 18^2 = 324$ $\quad$ $\text{gcf}(12, 324) = 12$

So we can eliminate choices B, C, and E.

Now let's try $a = 9$ and $b = 3$. Then $k = 3$. Let's check the answer choices that still remain.

A. $a + b = 9 + 3 = 12$ $\quad$ $\text{gcf}(9, 12) = 3$
D. $3b = 3 \cdot 3 = 9$ $\quad$ $\text{gcf}(9, 9) = 9$

So we can eliminate choice D, and the answer is A.

Notes: (1) If you are having trouble computing any of the gcfs in the solution to this problem, you may want to review the solutions and notes to problem 37.

(2) A rigorous approach to solving this problem is quite sophisticated and so I omit it here.

134. If x and y are integers and $x^2y + xy^2 + x^2y^2$ is odd, which of the following statements must be true? Indicate <u>all</u> such statements.

A. x is odd
B. xy is odd
C. $x + y + xy$ is odd

* Note that $x^2y + xy^2 + x^2y^2 = xy(x + y + xy)$. The only way a product can be odd is if each factor is odd. So, x, y and $x + y + xy$ all must be odd. Since the product of two odd integers is odd, xy must also be odd. So all three choices are answers: A, B, and C.

135. How many positive integers less than 5,000 are multiples of 7 and are equal to 11 times an even integer?

* Note that 11 times an even integer is just a multiple of $11 \cdot 2 = 22$. So we are looking for positive integers less than 5,000 that are multiples of both 7 and 22.

Since 7 and 22 have no prime factors in common, we are just looking for multiples of $7 \cdot 22 = 154$ that are less than 5000. The answer is just the integer part of $\frac{5000}{154} \approx 32.4675$. So we grid in $\mathbf{32}$.

136. If n is a positive integer such that the units (ones) digit of $n^2 + 4n$ is 7 and the units digit of n is <u>not</u> 7, what is the units digit of $n + 3$?

* By plugging in values of n, we find that for $n = 9$,

$$n^2 + 4n = 9^2 + 4 \cdot 9 = 81 + 36 = 117.$$

So $n = 9$ works, and $n + 3 = 9 + 3 = 12$. Therefore, the units digit of $n + 3$ is $\mathbf{2}$.

Solution showing the independence of n: $n^2 + 4n = n(n + 4)$. So we are looking at positive integers 4 units apart whose product ends in 7. Since 7 is odd, n must be odd. So n must end in 1, 3, 5, or 9. Note that we skip $n = 7$ since the problem forbids us from using it.

If n ends in 1, then $n + 4$ ends in 5, and $n(n + 4)$ ends in 5.
If n ends in 3, then $n + 4$ ends in 7, and $n(n + 4)$ ends in 1.
If n ends in 5, then $n + 4$ ends in 9, and $n(n + 4)$ ends in 5.
If n ends in 9, then $n + 4$ ends in 13, and $n(n + 4)$ ends in $\mathbf{7}$.

So n ends in a 9, and $n + 3$ ends in a $\mathbf{2}$.

LEVEL 5: ALGEBRA

$$0 < c < 1$$

137. Quantity A: $\frac{c-1}{c+1}$

Quantity B: $\frac{1-c}{c-1}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Solution by picking numbers: Let's try $c = 0.5$. Then we have $\frac{c-1}{c+1} = \frac{0.5-1}{0.5+1} = \frac{-0.5}{1.5} \approx -0.33$ and $\frac{1-c}{c-1} = \frac{1-0.5}{0.5-1} = \frac{0.5}{-0.5} = -1$. So Quantity A is greater than Quantity B. This narrows down the answer to A or D.

Let's try $c = 0.1$ next. Then $\frac{c-1}{c+1} = \frac{0.1-1}{0.1+1} = \frac{-0.9}{1.1} \approx -0.82$. We also have $\frac{1-c}{c-1} = \frac{1-0.1}{0.1-1} = \frac{0.9}{-0.9} = -1$. So Quantity A is greater than Quantity B again.

Let's try $c = 0.9$ as well. Then $\frac{c-1}{c+1} = \frac{0.9-1}{0.9+1} = \frac{-0.1}{1.9} \approx -0.05$. We also have $\frac{1-c}{c-1} = \frac{1-0.9}{0.9-1} = \frac{0.1}{-0.1} = -1$. So once again Quantity A is greater than Quantity B.

The evidence seems to suggest that the answer is choice A.

Notes: (1) Be careful when comparing negative numbers. For example, -0.33 is actually greater that -1, whereas 1 is greater than 0.33.

(2) There is no guarantee that this method will give us the right answer. No matter how many numbers we test, there is still always a small possibility that a number we didn't choose could produce a different result (leading to choice D being the answer).

(3) We minimized our risk of choosing the wrong answer by testing 3 different values for c: one right in the middle of 0 and 1, one close to 0, and once close to 1.

*** Algebraic solution:** First observe that $1 - c = -c + 1 = -(c - 1)$. It follows that Quantity B is always $\frac{1-c}{c-1} = \frac{-(c-1)}{(c-1)} = -1$.

So we want to know the relationship between $\frac{c-1}{c+1}$ and -1. (In other words, is $\frac{c-1}{c+1}$ less than, greater than, or equal to -1?)

Let's compare the two quantities using a ? in place of the unknown symbol (which is $<$, $>$, or $=$).

$$\frac{c-1}{c+1} \ ? \ -1$$
$$(c-1) \ ? -(c+1)$$
$$c - 1 \ ? \ -c - 1$$
$$c \ ? \ -c$$
$$2c \ ? \ 0$$
$$c \ ? \ 0$$

Since we were given $c > 0$ in the problem, we now know that the question mark is "$>$."

It follows that Quantity A is greater than Quantity B, choice A.

On the number line, the distance between b and 1 is 5.

138. Quantity A: The distance between b and 3 on the number line
Quantity B: 3

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

***** b can be either -4 or 6.

If $b = -4$, then the distance between b and 3 is 7, in which case Quantity A is greater than Quantity B.

If $b = 6$, then the distance between b and 3 is 3, in which case Quantity A is equal to Quantity B.

So the answer is D.

Notes: (1) "The distance between b and 1 is 5" can be expressed mathematically as $|b - 1| = 5$

This equation is equivalent to the following two equations without absolute values.

$$b - 1 = -5 \qquad \text{or} \qquad b - 1 = 5$$

The solution to the first equation is $b = -5 + 1 = -4$, and the solution to the second equation is $b = 5 + 1 = 6$.

(2) The distance between -4 and 3 is $|-4 - 3| = |-7| = 7$.

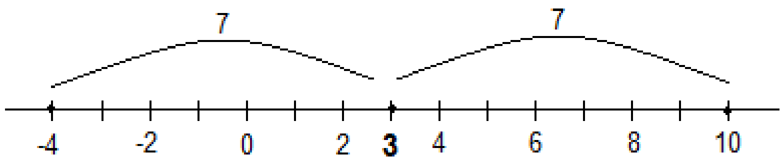
(3) The distance between 6 and 3 is $|6 - 3| = |3| = 3$.

Absolute Value and Distance: The **absolute value** of x, written $|x|$ is simply x if x is nonnegative, and $-x$ if x is negative. Put simply, $|x|$ just removes the minus sign if one is there.

Geometrically, $|x - y|$ is the distance between x and y. In particular, $|x - y| = |y - x|$.

Examples: $|5-3| = |3-5| = 2$ because the distance between 3 and 5 is 2.

If $|x-3| = 7$, then the distance between x and 3 is 7. So there are two possible values for x. They are $3+7=10$, and $3-7=-4$. See the figure below for clarification.



If $|x-3| < 7$, then the distance between x and 3 is less than 7. If you look at the above figure you should be able to see that this is all x satisfying $-4 < x < 10$.

If $|x-3| > 7$, then the distance between x and 3 is greater than 7. If you look at the above figure you should be able to see that this is all x satisfying $x < -4$ or $x > 10$.

Algebraically, we have the following. For $c > 0$,

$|x| = c$ is equivalent to $x = c$ or $x = -c$

$|x| < c$ is equivalent to $-c < x < c$

$|x| > c$ is equivalent to $x < -c$ or $x > c$.

Let's look at the same examples as before algebraically.

Examples: If $|x-3| = 7$, then $x-3=7$ or $x-3=-7$. So $x=10$ or $x=-4$.

If $|x-3| < 7$, then $-7 < x-3 < 7$. So $-4 < x < 10$.

If $|x-3| > 7$, then $x-3 < -7$ or $x-3 > 7$. So $x < -4$ or $x > 10$.

139. A girl scout has k boxes, each containing 10 cookies. After visiting p parents and selling c cookies to each of them, she has d cookies remaining. Which of the following expresses p in terms of k, c, and d ?

A. $\frac{10k-d}{c}$

B. $\frac{10k+d}{c}$

C. $\frac{10k}{c} - d$

D. $\frac{10c-d}{k}$

E. $\frac{10c+d}{k}$

Solution by picking numbers: Let's let $k = 2$, $\boldsymbol{p = 3}$, and $c = 4$. So the girl scout has $2 \cdot 10 = 20$ cookies. She sells 4 cookies to each of 3 parents for a total of $4 \cdot 3 = 12$ cookies. It follows that she has $20 - 12 = 8$ left. So $d = 8$.

Since we are trying to find p, we put a nice big, dark circle around $p = \mathbf{3}$. We now check each answer choice.

A. $\frac{10k-d}{c} = \frac{20-8}{4} = \frac{12}{4} = 3$

B. $\frac{10k+d}{c} = \frac{20+8}{4} = \frac{28}{4} = 7$

C. $\frac{10k}{c} - d = \frac{20}{4} - 8 = 5 - 8 = -3$

D. $\frac{10c-d}{k} = \frac{40-8}{2} = \frac{32}{2} = 16$

E. $\frac{10c+d}{k} = \frac{40+8}{2} = \frac{48}{2} = 24$

Since choices B, C, D, and E came out incorrect we can eliminate them. So the answer is choice A.

Important note: It is very important that you check every answer choice when picking numbers. As we have seen in problem 38, specific numbers can lead to more than one choice coming out correct.

*** Algebraic solution:** The girl scout has a total of $k \cdot 10 = 10k$ cookies (k boxes with 10 cookies per box). She sells a total of pc cookies (p parents and c cookies per parent). So the number of cookies she has remaining is $d = 10k - pc$.

We need to solve this last equation for p. We subtract $10k$ from each side of the equation to get $d - 10k = -pc$. We now divide each side of this last equation by $-c$ to get $p = \frac{d-10k}{-c} = \frac{-(d-10k)}{c} = \frac{10k-d}{c}$, choice A.

Note: To avoid all of the minus signs, we can also solve the equation $d = 10k - pc$ by adding pc to each side of the equation while simultaneously subtracting d from each side of the equation to get $pc = 10k - d$. We then just divide by c to get $p = \frac{10k-d}{c}$.

140. If $x < 0$ and $0 < y < |x|$, which of the following must be true ?

A. $y^2 - x^2 < 0$
B. $x^2 - y^2 < 0$
C. $(y - x)^2 < 0$
D. $(x - y)^2 < 0$
E. $(x + y)^2 < 0$

Solution by picking numbers: Let's let $x = -3$ and $y = 2$. Then we have $x < 0$, and $|x| = |-3| = 3$. So $0 < y < |x|$. Let's check each answer choice.

A. $y^2 - x^2 = 2^2 - (-3)^2 = 4 - 9 = -5 < 0$
B. $x^2 - y^2 = (-3)^2 - 2^2 = 9 - 4 = 5 > 0$
C. $(y - x)^2 = \left(2 - (-3)\right)^2 = (2 + 3)^2 = 5^2 = 25 > 0$
D. $(x - y)^2 = (-3 - 2)^2 = (-5)^2 = 25 > 0$
E. $(x + y)^2 = (-3 + 2)^2 = (-1)^2 = 1 > 0$

Since choices B, C, D, and E came out nonnegative, we can eliminate them. So the answer is choice A.

Notes: (1) It is very important that you check every answer choice when picking numbers. As we have seen in problem 38, specific numbers can lead to more than one choice coming out correct.

(2) Students often confuse $(-3)^2$ and -3^2. Let's do these computations.

$$(-3)^2 = (-3)(-3) = 9$$
$$-3^2 = (-1) \cdot 3^2 = (-1) \cdot 9 = -9$$

(3) We do not need to finish all of the computations when checking the answer choices. As soon as we see that the result will not be negative we can stop. For example, in B, once we have $9 - 4$, we know we are going to get a positive number, and so we can eliminate choice B.

(4) When any real number is squared, the result is nonnegative. So we can immediately eliminate C, D, and E without actually having to do any computations.

*** Algebraic solution:** We square each side of the inequality $y < |x|$ to get $y^2 < |x|^2$, or equivalently $y^2 < x^2$. We now subtract x^2 from each side of this last inequality to get $y^2 - x^2 < 0$, choice A.

Notes: (1) If a and b are nonnegative, then the inequalities $a < b$ and $a^2 < b^2$ are equivalent. In this problem, we have $a = y$ and $b = |x|$. So $y < |x|$ is equivalent to $y^2 < |x|^2$.

(2) This is actually trickier than it seems. In Note (1) it is essential that a and b be nonnegative. For example, if $a = -1$ and $b = 1$, then $a < b$, but $a^2 = b^2 = 1$. So $a^2 \nless b^2$.

In this problem we are given $0 < y$. Also $|x|$ is always nonnegative. So we can use the result from Note (1).

(3) If x is a real number, then x^2 and $|x|^2$ are always equal. To convince yourself, plug a few different numbers (positive, negative and zero) into both of these expressions.

141. If $\frac{1}{3^a} = \frac{1}{3^b} + \frac{1}{3^b} + \frac{1}{3^b}$, then a expressed in terms of b is

A. $\frac{b}{3}$
B. $b - 1$
C. $b + 1$
D. $3b$
E. b^3

Solution by picking a number: Let's choose a value for b, say $b = 2$. Then we have $\frac{1}{3^a} = \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3^2} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$. So $a = \mathbf{1}$. Put a nice big, dark circle around $a = \mathbf{1}$. We now substitute $b = 2$ into each answer choice.

A. $\frac{b}{3} = \frac{2}{3}$
B. $b - 1 = 2 - 1 = 1$
C. $b + 1 = 2 + 1 = 3$
D. $3b = 3 \cdot 2 = 6$
E. $b^3 = 2^3 = 8$

Since choices A, C, D, and E did not come out to -1, we can eliminate them. So the answer is choice B.

*** Algebraic solution:** $\frac{1}{3^b} + \frac{1}{3^b} + \frac{1}{3^b} = 3 \cdot \frac{1}{3^b} = \frac{3^1}{3^b} = \frac{1}{3^{b-1}}$.

So we have $\frac{1}{3^a} = \frac{1}{3^{b-1}}$, and it follows that $a = b - 1$, choice B.

Notes: (1) Adding the same number 3 times is equivalent to multiplying that number by 3. In other words, $x + x + x = 3x$. If we replace x by $\frac{1}{3^b}$, we get $\frac{1}{3^b} + \frac{1}{3^b} + \frac{1}{3^b} = 3 \cdot \frac{1}{3^b}$

(2) $x \cdot \frac{1}{y} = \frac{x}{y}$. Replacing x by 3, and y by 3^b gives us $3 \cdot \frac{1}{3^b} = \frac{3}{3^b}$.

(3) $3 = 3^1$ so that $\frac{3}{3^b} = \frac{3^1}{3^b}$.

(4) When we divide expressions with the same base we subtract the exponents: $\frac{a^x}{a^y} = a^{x-y}$, or equivalently $\frac{a^x}{a^y} = \frac{1}{a^{y-x}}$

Notice that if we compute $x - y$, then the exponent stays in the numerator, whereas if we compute $y - x$, the exponent stays in the denominator.

So we have $\frac{3^1}{3^b} = 3^{1-b}$, or equivalently $\frac{3^1}{3^b} = \frac{1}{3^{b-1}}$.

In the above solution I chose to use the second one.

(5) Once we have $\frac{1}{3^a} = \frac{1}{3^{b-1}}$ we can set the exponents equal. That is we have $a = b - 1$.

(6) We could have also chosen to write $\frac{3^1}{3^b} = 3^{1-b}$. In this case we have $\frac{1}{3^a} = 3^{1-b}$. We can rewrite $\frac{1}{3^a}$ as 3^{-a}. So we have $3^{-a} = 3^{1-b}$. It follows that $-a = 1 - b$, and so $a = -(1 - b) = -1 + b = b - 1$.

142. If $(7^{4m})(343) = 7^k$, where k and m are integers, what is the value of m in terms of k ?

A. $\frac{k-3}{4}$
B. $\frac{k-4}{3}$
C. $3k - 4$
D. $4k - 3$
E. $4k + 3$

*** Algebraic solution:** We have $343 = 7^3$, and so it follows that $(7^{4m})(343) = (7^{4m})(7^3) = 7^{4m+3}$.

So we have $7^{4m+3} = 7^k$, and therefore $4m + 3 = k$.

We solve this last equation for m. We subtract 3 from each side of this equation to get $4m = k - 3$. We then divide each side of this last equation by 4 to get $m = \frac{k-3}{4}$, choice A.

Notes: (1) See problem 43 for a review of the laws of exponents used here.

(2) This problem can also be solved by picking numbers. I leave the details to the reader.

143. If $t^{-2} - 4t^{-1} - 12 = 0$, which of the following could be the value of t ? Indicate <u>all</u> such values.

A. -6
B. -2
C. $-\frac{1}{2}$
D. $-\frac{1}{6}$
E. $\frac{1}{6}$
F. $\frac{1}{2}$
G. 2
H. 6

Solution using a substitution: We make the substitution $u = t^{-1}$. It follows that $u^2 = (t^{-1})^2 = t^{-2}$. So the given equation becomes

$$u^2 - 4u - 12 = 0.$$

We can solve this equation in several different ways. The quickest way in this case is by factoring.

$$(u - 6)(u + 2) = 0$$
$$u - 6 = 0 \quad \text{or} \quad u + 2 = 0$$
$$u = 6 \quad \text{or} \quad u = -2$$

We now replace u by t^{-1} and solve for t.

$$t^{-1} = 6 \quad \text{or} \quad t^{-1} = -2$$
$$t = \frac{1}{6} \quad \text{or} \quad t = -\frac{1}{2}$$

So the answers are C and E.

Notes: (1) We can solve the quadratic equation $u^2 - 4u - 12 = 0$ in several other ways. Here are two other methods:

Quadratic formula: We identify $a = 1$, $b = -4$, and $c = -12$.

$$\boldsymbol{u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16+48}}{2} = \frac{4 \pm \sqrt{64}}{2} = \frac{4 \pm 8}{2} = 2 \pm 4.}$$

So we get $u = 2 + 4 = 6$ or $u = 2 - 4 = -2$.

Completing the square: For this solution we move the constant to the right hand side to get $u^2 - 4u = 12$.

We take half of -4, which is -2, and square this number to get 4. We then add 4 to each side of the equation to get $u^2 - 4u + 4 = 12 + 4$. This is equivalent to $(u - 2)^2 = 16$. We now apply the square root property to get $u - 2 = \pm 4$. So $u = 2 \pm 4$. This yields the solutions $2 + 4 = 6$, and $2 - 4 = -2$.

(2) Once we find u, we need to remember to replace u by t^{-1}.

(3) Recall that $t^{-1} = \frac{1}{t^1} = \frac{1}{t}$. So we can solve the equation $t^{-1} = 6$ by taking the reciprocal of each side of this equation. We get $t = \frac{1}{6}$.

Similarly, the equation $t^{-1} = -2$ has solution $t = \frac{1}{-2} = -\frac{1}{2}$.

*** Solution without a formal substitution:** We can factor the left hand side of the given equation as $t^{-2} - 4t^{-1} - 12 = (t^{-1} - 6)(t^{-1} + 2)$.

We set each factor equal to 0 to get $t^{-1} - 6 = 0$ or $t^{-1} + 2 = 0$.

So $t^{-1} = 6$ or $t^{-1} = -2$. Therefore, $t = \frac{1}{6}$ or $t = -\frac{1}{2}$.

The answers are C and E.

144. Each day, a factory's total expenses are equal to a fixed daily expense plus a variable expense that is directly proportional to the number of units of product produced by the factory during that day. If the factory's total expenses for a day in which it produces 3,000 units are \$5,500, and the total expenses for a day in which it produces 7,000 units are \$8,200, what is the factory's fixed daily expense? (Disregard the dollar sign when gridding your answer.)

*** Solution using the point-slope form of the equation of a line:** The paragraph is describing a linear equation whose graph passes through the points $(3{,}000, 5{,}500)$ and $(7{,}000, 8{,}200)$.

We use these two points to compute the slope of the line:

$$m = \frac{8{,}200 - 5{,}500}{7{,}000 - 3{,}000} = \frac{2{,}700}{4{,}000} = \frac{27}{40}$$

We now use the slope and either of the points to write an equation of the line in point-slope form. Let's use the point $(3{,}000, 5{,}500)$.

$$y - 5{,}500 = \frac{27}{40}(x - 3{,}000)$$

We now rewrite this equation in slope-intercept form.

$$y - 5{,}500 = \frac{27}{40}x - 2{,}025$$
$$y = \frac{27}{40}x + 3{,}475$$

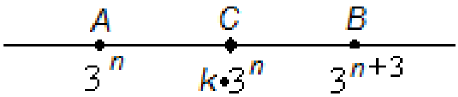
The fixed daily expense is \$3,475. So the answer is **3475.**

Notes: (1) We have $y = mx + b$, where b is the factory's fixed daily expense, m is the factory's variable expense, x is the number of units produced by the factory during the given day, and y is the factory's total expense for the day.

(2) The **point-slope form of an equation of a line** is $\boldsymbol{y - y_0 = m(x - x_0)}$ where $\boldsymbol{m}$ is the slope of the line and $(\boldsymbol{x_0, y_0})$ is any point on the line.

In this problem we used the point $(3{,}000, 5{,}500)$ and we found that the slope was $m = \frac{27}{40}$.

LEVEL 5: GEOMETRY



Point C is the midpoint of $\overline{AB}$ and n is a positive integer.

145. Quantity A: k
Quantity B: 14

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

Solution by picking numbers: Let's let n = 1. Then the 3 numbers from left to right are 3, $3k$ and 81. So $3k$ should be the average of 3 and 81.

$$3k = \frac{3 + 81}{2} = \frac{84}{2} = 42.$$

Thus, $k = \frac{42}{3} = 14$. So in this case the Quantities are equal.

Now let's try $n = 2$. Then the 3 numbers from left to right are 9, $9k$ and 243. So $9k$ should be the average of 9 and 243.

$$9k = \frac{9 + 243}{2} = \frac{252}{2} = 126.$$

Thus, $k = \frac{126}{9} = 14$. So again the Quantities are equal.

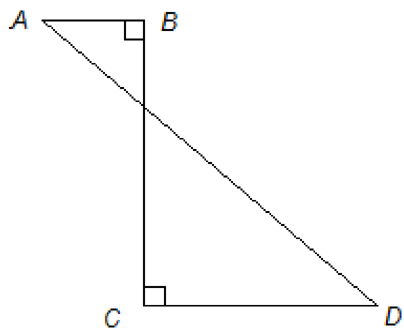
The evidence seems to suggest that the answer is choice C.

Note: There is no guarantee with this method of solution that we would get the right answer. Although it seems likely that k will always come out to 14, there could be a positive integer n for which it does not. In other words, all we really did was narrow down the possible answers to C or D.

*** Complete solution:** $k \cdot 3^n$ should be the average of 3^n and 3^{n+3}. So we have

$$k \cdot 3^n = \frac{3^n + 3^{n+3}}{2} = \frac{3^n + 3^n 3^3}{2} = 3^n(\frac{1+27}{2}) = 3^n\left(\frac{28}{2}\right) = 14 \cdot 3^n$$

Thus, we have $k = 14$. So the Quantities are equal and the answer is C.

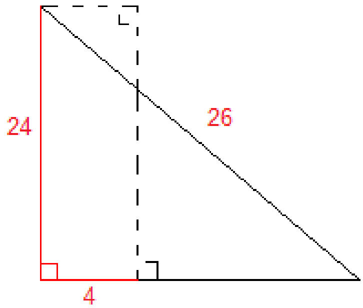


$AB = 4$, $BC = 24$, and $AD = 26$

146. Quantity A: The length of segment $\overline{CD}$
Quantity B 10

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

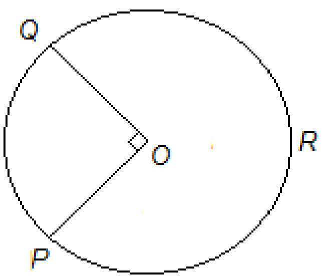
*** Solution by moving the sides of the figure around:** The problem becomes much simpler if we "move" $\overline{BC}$ to the left and $\overline{AB}$ to the bottom as shown below.



We now have a single right triangle and we can either use the Pythagorean Theorem, or better yet notice that $26 = 13 \cdot 2$ and $24 = 12 \cdot 2$. Thus, the other leg of the triangle is $5 \cdot 2 = 10$. So we see that $\overline{CD}$ must have length $10 - 4 = 6$. So Quantity B is greater than Quantity A, choice B.

Note: If we didn't notice that this was a multiple of a $5 - 12 - 13$ triangle, then we would use the Pythagorean Theorem as follows.

$$(x + 4)^2 + 24^2 = 26^2$$
$$(x + 4)^2 + 576 = 676$$
$$(x + 4)^2 = 100$$
$$x + 4 = 10$$
$$x = 6$$



147. In the figure above, the circle has center O and radius 8. What is the length of arc PRQ?

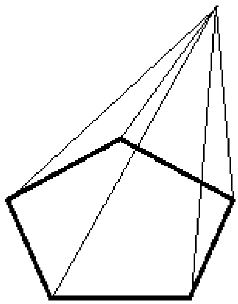
A. 12π
B. $24\sqrt{2}$
C. 6π
D. $12\sqrt{2}$
E. $3\pi\sqrt{2}$

Solution using a ratio: Note that arc PRQ measures $270°$ and the circumference of the circle is $C = 2\pi r = 16\pi$. So we solve for s in the following ratio.

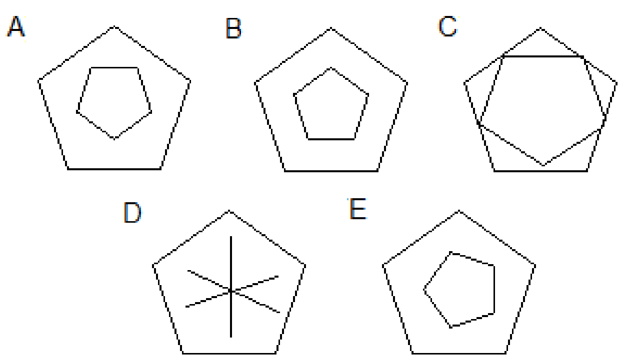
$$\frac{270}{360} = \frac{s}{16\pi}$$

Cross multiplying gives $360s = 4320\pi$, and therefore $s = \frac{4320\pi}{360} = 12\pi$, choice A.

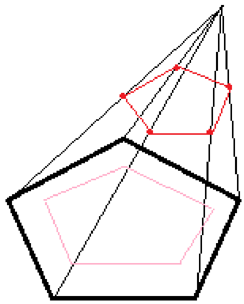
*** Quick solution:** Note that arc PRQ is $\frac{3}{4}$ of the circumference of the circle and therefore PRQ has length $s = (\frac{3}{4})(16\pi) = 12\pi$, choice A.



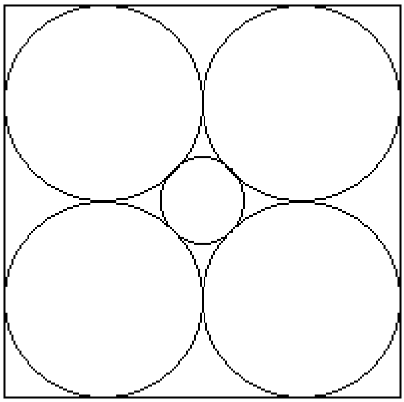
148. The figure above is a pyramid with four isosceles triangular faces and a base that is a regular pentagon. Points A, B, C, D and E (not shown) are the midpoints of the edges that are not in the plane of the base. Line segments are to be drawn on the triangular faces such that each segment connects two of these points. Which of the following is a representation of how these line segments could appear if viewed through the pentagonal base?



* The following picture illustrates the solution.



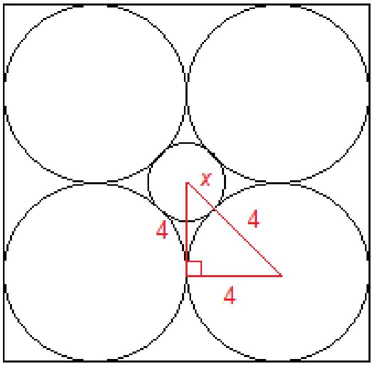
We have plotted points at the midpoint of each edge not in the plane of the base, then attached them with line segments drawn on the triangular faces. Finally, we lightly sketched the projection of the resulting pentagon onto the base. We see that the answer is choice B.



149. In the figure above, each of the four large circles is tangent to two of the other large circles, the small circle, and two sides of the square. If the radius of each of the large circles is 4, what is the diameter of the small circle?

A. $\sqrt{2}$
B. 1
C. $8\sqrt{2}-8$
D. $4\sqrt{2}-4$
E. $\sqrt{2}-1$

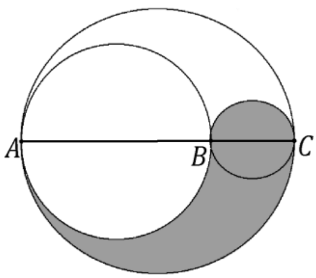
* We draw an isosceles right triangle.



Note that each length labeled with a 4 is equal to the radius of one of the larger circles (the radius is half the diameter). The length labeled x is the radius of the smaller circle. An isosceles right triangle is the same as a 45, 45, 90 right triangle. By looking at the formula for a 45, 45, 90 triangle we see that $x + 4 = 4\sqrt{2}$ and so $x = 4\sqrt{2} - 4$. The diameter is then $2x = 2(4\sqrt{2} - 4) = 8\sqrt{2} - 8$, choice C.

Notes: (1) We can also use the Pythagorean Theorem to find x. We have $(x + 4)^2 = 4^2 + 4^2 = 16 + 16 = 32$. So $x + 4 = \sqrt{32} = 4\sqrt{2}$ and so $x = 4\sqrt{2} - 4$, choice C.

(2) If you are not comfortable simplifying square roots, you can use the calculator to estimate them.



150. $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are diameters of the three circles shown above. If $BC = 4$ and $AB = 5BC$, what is the area of the shaded region?

A. 48π
B. 24π
C. 12π
D. 6π
E. 3π

* We first find the radius of each of the three circles. The diameter of the small circle is 4, and so its radius is 2. The diameter of the medium-sized circle is $5 \cdot 4 = 20$, and so its radius is 10. The diameter of the largest circle is $20 + 4 = 24$, and so its radius is 12. We can now find the area of the shaded region as follows.

$$\begin{aligned} A &= \tfrac{1}{2}(\text{Area of big circle}) - \tfrac{1}{2}(\text{Area of medium circle}) + \tfrac{1}{2}(\text{Area of small circle}) \\ &= \frac{1}{2}(\pi \cdot 12^2) - \frac{1}{2}(\pi \cdot 10^2) + \frac{1}{2}(\pi \cdot 2^2) \\ &= \frac{1}{2}(\pi \cdot 144) - \frac{1}{2}(\pi \cdot 100) + \frac{1}{2}(\pi \cdot 4) \\ &= \frac{1}{2} \cdot 48\pi \\ &= 24\pi \end{aligned}$$

Thus, the answer is choice B.

151. Points Q, R and S lie in a plane. If the distance between Q and R is 18 and the distance between R and S is 11, which of the following values could be the distance between Q and S? Indicate <u>all</u> such values.

A. 6
B. 7
C. 11
D. 13
E. 28
F. 29
G. 30

*** Solution using the triangle rule:** If Q, R and S form a triangle, then the length of QS is between $18 - 11 = 7$ and $18 + 11 = 29$. The extreme cases 7 and 29 form straight lines. In this problem that is fine, so the distance between Q and S is between 7 and 29, inclusive. So the answers are B, C, D, E, and F.

Note: The triangle rule states that the length of the third side of a triangle is between the sum and difference of the lengths of the other two sides.

152. The cost to maintain a certain type of garden is \$10 per 15 square feet for one month. Grace has a garden of this type that measures 2 feet by 30 feet, and Ashley has a garden of this type that measures 40 feet by 6 feet. How much more does Ashley pay than Grace <u>each year</u> to maintain her garden?

* The number of square feet in Grace's garden is $2 \cdot 30 = 60$. So it costs her $4 \cdot 10 = 40$ dollars per month. The number of square feet in Ashley's garden is $40 \cdot 6 = 240$. So it costs her $16 \cdot 10 = 160$ dollars per month. So Ashley pays $160 - 40 = 120$ more per month than Grace. So each year, Ashley pays $12 \cdot 120 = \mathbf{1440}$ more.

Note: $\frac{60}{15} = 4$ and $\frac{240}{15} = 16$. I am dividing these numbers by 15 because the cost is $10 per 15 square feet.

Level 5: Data Analysis

$$f = a + b + c + d + e$$

153. Quantity A: The average (arithmetic mean) of a, b, c, d, e, and f in terms of f

Quantity B: $\frac{f}{5}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* The average of a, b, c, d, e, and f is

$$\frac{a+b+c+d+e+f}{6}$$
$$= \frac{a+b+c+d+e+a+b+c+d+e}{6}$$
$$= \frac{2a+2b+2c+2d+2e}{6}$$
$$= \frac{2(a+b+c+d+e)}{6}$$
$$= \frac{2f}{6}$$
$$= \frac{f}{3}$$

Since $\frac{f}{3} > \frac{f}{5}$, the answer is A.

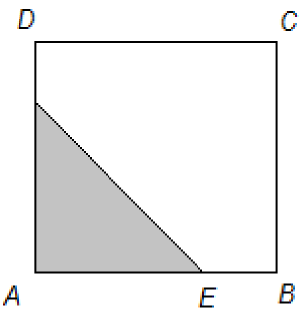
Notes: (1) Recall that if the numerator of a positive fraction stays the same, then increasing the denominator makes the value of the fraction smaller.

(2) We can pick numbers to eliminate answer choices. For example, let's let $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 5$. Then $f = 15$, and the average of a, b, c, d, e, and f is

$$\frac{1 + 2 + 3 + 4 + 5 + 15}{6} = \frac{30}{6} = 5$$

We also have $\frac{f}{5} = \frac{15}{5} = 3$.

So in this case Quantity A is greater than Quantity B. So the answer is either choice A or D.



$ABCD$ is a square, $AD = 6$, $EB = 6 - 2b$, the triangle is isosceles, and the probability that a randomly selected point in square $ABCD$ will be in the shaded triangle is $\frac{1}{3}$.

154. Quantity A: b
Quantity B: 2

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

* $AE = 6 - (6 - 2b) = 6 - 6 + 2b = 2b$. So the area of the triangle is

$$\frac{1}{2}(2b)(2b) = 2b^2.$$

The area of the square is $6 \cdot 6 = 36$. Thus, the probability of choosing a point in the triangle is $\frac{2b^2}{36} = \frac{b^2}{18}$. We are given that this is equal to $\frac{1}{3}$. So we have

$$\frac{b^2}{18} = \frac{1}{3}$$

We cross multiply and divide to get

$3b^2 = 18$. So $b^2 = 6$, and $b = \sqrt{6}$.

$2 = \sqrt{4} < \sqrt{6}$. So Quantity A is greater than Quantity B, choice A.

$$\frac{1}{x^3}, \frac{1}{x^2}, \frac{1}{x}, x, x^2, x^3$$

155. If $-1 < x < 0$, what is the median of the six numbers in the list above?

A. $\frac{1}{x}$

B. x^2

C. $\frac{x^2(x+1)}{2}$

D. $\frac{x(x^2+1)}{2}$

E. $\frac{x^2+1}{2x}$

*** Solution by picking a number:** Let's choose $x = -0.5$.

We use the calculator to compute the given expressions.

$$\frac{1}{x^3} = -8 \quad \frac{1}{x^2} = 4 \quad \frac{1}{x} = -2 \quad x = -0.5 \quad x^2 = 0.25 \quad x^3 = -0.125$$

Now let's place them in increasing order.

$$-8, -2, -0.5, -0.125, 0.25, 4$$

The median is the average of -0.5 and -0.125, i.e., it is $\mathbf{-0.3125}$. Now let's substitute $x = -0.5$ into each answer choice.

A. -2
B. 0.25
C. 0.0625
D. -0.3125
E. -1.25

Since choices A, B, C, and E came out incorrect, we can eliminate them. So the answer is choice D.

Important note: D is **not** the correct answer simply because it is equal to -0.3125. It is correct because all four of the other choices are **not** -0.3125. **You absolutely must check all five choices!**

156. The integers 1 through 5 are written on each of five cards. The cards are shuffled and one card is drawn at random. That card is then replaced, the cards are shuffled again and another card is drawn at random. This procedure is repeated one more time (for a total of three times). What is the probability that the sum of the numbers on the three cards drawn was 14 or 15?

A. $\frac{1}{125}$
B. $\frac{2}{125}$
C. $\frac{4}{125}$
D. $\frac{1}{25}$
E. $\frac{2}{25}$

* The total number of possibilities for the three cards is $5 \cdot 5 \cdot 5 = 125$.

$$5 + 5 + 5 = 15$$
$$5 + 5 + 4 = 14$$
$$5 + 4 + 5 = 14$$
$$4 + 5 + 5 = 14$$

Thus, there are 4 possibilities that give the desired sum. The probability is therefore $\frac{4}{125}$, choice C.

157. A five digit number is to be formed using each of the digits 1, 2, 3, 4, and 5 exactly once. How many such numbers are there in which the digits 1 and 2 are not next to each other?

A. 36
B. 48
C. 60
D. 72
E. 120

Let's start by thinking about where 1 can go. There are 2 cases:

1st Case: 1 is placed at an end. In this case, there are now 3 places where the 2 can go. After the 2 is placed, there are 3 places for the 3, then 2 places for the 4, and then 1 place for the 5. By the counting principle there are $2 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = \mathbf{36}$ ways to form the five digit number when the 1 is placed at either end (note that the first 2 comes from the fact that we have 2 choices for 1 – the far left or the far right).

2nd Case: 1 is not placed at an end. In this case, there are now 2 places where the 2 can go, and then the rest is the same as case 1. So again by the counting principle there are $3 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = \mathbf{36}$ ways to form the five digit number when the 1 is **not** placed at either end (note that the first 3 comes from the fact that we have 3 choices for 1 – each of the 3 middle positions).

So adding up the possibilities from cases 1 and 2, we get 36 + 36 = 72 possibilities all together, choice D.

*** Quicker Method:** Let's first compute the number of ways to place the 1 and 2 with 1 to the left of 2. If the 1 is placed in the leftmost position, then there are 3 places to put the 2 to the right of the 1. If the 1 is placed in the next position to the right, then there are 2 places to put the 2 to the right of the 1. If the 1 is placed in the middle position, then there is 1 place to put the 2 to the right of the 1. So there are $3 + 2 + 1 = 6$ places to put the 1 and 2 with $1 < 2$. By symmetry, there are 6 places to put the 1 and the 2 with $2 < 1$. So all together there are 12 places to put the 1 and 2. Once the 1 and 2 are placed, there are 3 places to put the 3, then 2 places to put the 4, and 1 place to put the 5. By the counting principle the answer is $12 \cdot 3 \cdot 2 \cdot 1 = \mathbf{72}$.

$$2x, 6.1, 5.7, x, 2.3$$

158. The five numbers shown are listed in decreasing order. Which of the following values could be the range of the six numbers? Indicate all such values.

A. 2.3
B. 2.7
C. 3.5
D. 4.1
E. 8.3
F. 8.9
G. 9.2
H. 9.8

* **Algebraic solution:** We must have $2x \geq 6.1$ and $2.3 \leq x \leq 5.7$.

Multiplying the last inequalities by 2 gives $4.6 \leq 2x \leq 11.4$.

Combining both sets of inequalities above yields $6.1 \leq 2x \leq 11.4$

Subtracting 2.3 gives us $3.8 \leq 2x - 2.3 \leq 9.1$

So the range of the six numbers is between 3.8 and 9.1. Therefore, the answers are D, E, and F.

159. An experiment has five possible outcomes. The outcomes are mutually exclusive and have probabilities $x, \frac{x}{2}, \frac{x}{3}, \frac{x}{4}$, and $\frac{x}{6}$. What is the value of x? Express your answer as a fraction.

* The probabilities must add up to 1. So we have

$$x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 1$$

The least common denominator of the left hand side of the above equation is 12. So we multiply each side of the equation by 12 to get

$$12\left(x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6}\right) = 1(12)$$

$$12x + 12 \cdot \frac{x}{2} + 12 \cdot \frac{x}{3} + 12 \cdot \frac{x}{4} + 12 \cdot \frac{x}{6}) = 12$$

$$12x + 6x + 4x + 3x + 2x = 12$$

$$27x = 12$$

$$x = \frac{12}{27} = \frac{4}{9}$$

So we grid in $\mathbf{4/9}$.

160. For 5 numbers in a list of increasing numbers, the average (arithmetic mean), median, and mode are all equal to 11. The range of the list is 7. The second number in the list is less than 11 and 2 more than the least number in the list. What is the greatest number in the list? Express your answer as a fraction.

* Since the median and mode are both equal to 11, and the list is increasing, the numbers have one of the following forms:

$$x, y, 11, 11, z \qquad \text{or} \qquad x, 11, 11, y, z$$

Since the second number in the list is less than 11, we know that the form on the left is the correct one.

Since the second number in the list is 2 more than the least number in the list, we have $y = x + 2$. So the list has the following form:

$$x, x + 2, 11, 11, z$$

Since the range of the list is 7, we have $z - x = 7$, or equivalently, $z = x + 7$. So the list looks as follows:

$$x, x + 2, 11, 11, x + 7$$

Now, the average of the 5 numbers is 11, and so the sum of the 5 numbers is $5 \cdot 11 = 55$. So we have

$$x + (x + 2) + 11 + 11 + (x + 7) = 55$$
$$3x + 31 = 55$$
$$3x = 24$$
$$x = 8$$

So the greatest number in the list is $x + 7 = 8 + 7 = \mathbf{15}$.

Supplemental Problems Questions

Level 1: Arithmetic

1. Quantity A: $\frac{3}{5} - \frac{1}{3}$

   Quantity B: $\frac{2}{3} - \frac{2}{5}$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

2. Quantity A: The number of weeks in 30 years
   Quantity B: The number of seconds in 25 minutes

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

A supermarket purchased watermelons at a cost of \$2 per watermelon and sold each of them for 80 percent above cost.

3. Quantity A: The price at which the supermarket sold each watermelon
   Quantity B: \$3.50

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

4. Quantity A: -3^6
   Quantity B: $(-3)^6$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

5. What is 0.7215696 rounded to the nearest thousandth?

   A. 0.72
   B. 0.721
   C. 0.722
   D. 0.7215
   E. 0.7216

6. A positive integer is called a palindrome if it reads the same forward as it does backward. For example, 2442 is a palindrome. What is the smallest palindrome greater than 2513?

   A. 2112
   B. 2514
   C. 2525
   D. 2552
   E. 2553

7. If the sales tax on a mobile phone priced at $\$500$ is between 4 percent and 8 percent, then the cost (price plus sales tax) of the phone could be which of the following? Indicate <u>all</u> such values.

   A. $\$510$
   B. $\$525$
   C. $\$538$
   D. $\$545$
   E. $\$549$
   F. $\$560$

8. If 4.56 is rounded to the nearest tenth and the result is doubled, what is the final result?

LEVEL 1: ALGEBRA

$$x + \frac{2}{3} = 4 + \frac{5}{3}$$

9. Quantity A: x
   Quantity B: $6\frac{1}{3}$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

$$x < y < 0$$

10. Quantity A: $\frac{xy}{x}$
    Quantity B: $\frac{xy}{y}$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

$$a - 3 = 11$$
$$b - a = -6$$

11. Quantity A: b
    Quantity B: 8

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

$$y < 0$$

12. Quantity A: $y - 3$
    Quantity B: $3 - y$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

13. If Robert drove a miles in b hours, which of the following represents his average speed, in miles per hour?

    A. $\frac{a}{b}$
    B. $\frac{b}{a}$
    C. $\frac{1}{ab}$
    D. ab
    E. a^2b

14. If $1+x+2+x+3=x+1+x+2+x$, what is the value of x?

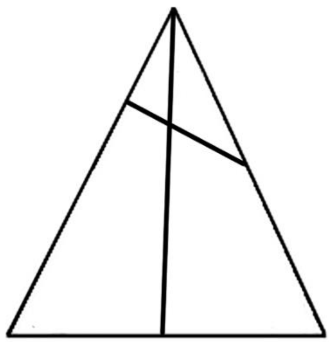
    A. 1
    B. 2
    C. 3
    D. 4
    E. 5

15. If $7(x-5)=6(x-4)$, which of the following could be the value of x? Indicate all such values.

    A. 8
    B. 9
    C. 10
    D. 11
    E. 12
    F. 15

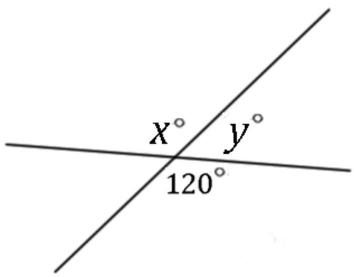
16. If $a = \frac{7}{b}$ and $c = 8a$, what is the value of c when $b = 42$? Express your answer as a fraction.

LEVEL 1: GEOMETRY



17. Quantity A: The total number of triangles shown above
    Quantity B: 5

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



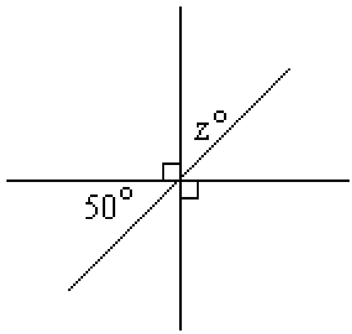
18. Quantity A: x
    Quantity B: $2y$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

A rectangular box is 5 feet long and 3 feet high and has a volume of 45 cubic feet.

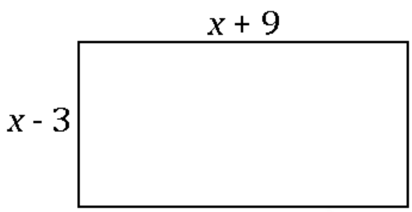
19. Quantity A: The width of the box
    Quantity B: 4 feet

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



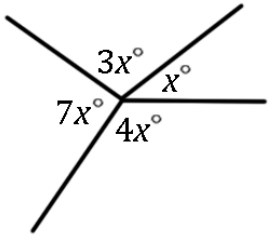
20. Quantity A: z
    Quantity B: 50

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



21. If the perimeter of the rectangle above is 80, what is the value of x?

    A. 20
    B. 19
    C. 18
    D. 17
    E. 16

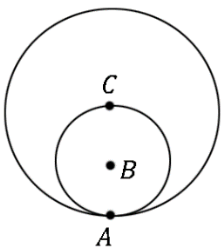


22. In the figure above, four line segments meet at a point to form four angles. What is the value of x?

   A. 18
   B. 24
   C. 30
   D. 40
   E. 60

23. Which of the following could be the perimeter of an isosceles triangle if one side has length 3 inches and another side has length 7 inches? Indicate <u>all</u> possible perimeters.

   A. 9 inches
   B. 11 inches
   C. 13 inches
   D. 15 inches
   E. 17 inches
   F. 19 inches



24. In the figure above, A, B, and C lie on the same line. B is the center of the smaller circle, and C is the center of the larger circle. If the radius of the smaller circle is 7, what is the diameter of the larger circle?

LEVEL 1: DATA ANALYSIS

There are exactly 13 coins in a bag: 4 pennies, 2 nickels, 3 dimes, and the rest are quarters. One coin is selected at random from the bag.

25. Quantity A: The probability that the selected coin is a quarter
Quantity B: The probability that the selected coin is a penny

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

The average (arithmetic mean) of four numbers is 100. Three of the numbers are 60, 70 and 140.

26. Quantity A: The fourth number
Quantity B: 120

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

List A: $2, 9, 4, 4, 18$
List B: $3, 12, 3, 3, 9$

27. Quantity A: The mode of the numbers in list A
Quantity B: The mode of the numbers in list B

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

20% of the marbles in a jar are green.

28. Quantity A: The total number of marbles in the jar

    Quantity B: 21

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

29. Joe, Mike, Phil, John and Fred own a total of 138 CDs. If John owns 38 of them, what is the average (arithmetic mean) number of CDs owned by Joe, Mike, Phil and Fred?

    A. 10
    B. 15
    C. 20
    D. 25
    E. 30

$$3, 4, 5, 6, 7, 8, 9, 10, 11$$

30. If a number is selected at random from the list above, what is the probability that it will be less than 7?

    A. $\frac{2}{9}$
    B. $\frac{1}{3}$
    C. $\frac{4}{9}$
    D. $\frac{2}{3}$
    E. 1

31. For which of the following lists of 5 numbers is the median (arithmetic mean) greater than the mode? Indicate <u>all</u> such lists.

    A. 1, 1, 3, 4, 5
    B. 1, 2, 3, 6, 6
    C. 1, 3, 3, 4, 5
    D. 2, 2, 3, 4, 5
    E. 1, 2, 3, 4, 4
    F. 1, 1, 3, 3, 3

32. In a jar there are exactly 56 marbles, each of which is yellow, purple, or blue. The probability of randomly selecting a yellow marble from the jar is $\frac{2}{7}$, and the probability of randomly selecting a purple marble from the jar is $\frac{3}{7}$. How many marbles in the jar are blue?

LEVEL 2: ARITHMETIC

k is a positive integer

$$\frac{5}{3} = \frac{m}{k}$$

33. Quantity A: m
Quantity B: 5

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$-1 < x < y < 0$$

34. Quantity A: $x + y$
Quantity B: $x^2 + y^2$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

m and n are nonnegative integers, $m > 2$, $n < 3$

35. Quantity A: m
Quantity B: $1.5n$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

36. Quantity A: 490,000
Quantity B: $(699.99)^2$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

37. A room has 1200 square feet of surface that needs to be painted. If 2 gallons of paint will cover 450 square feet, what is the least whole number of gallons that must be purchased in order to have enough paint to cover the entire surface?

A. 2
B. 3
C. 4
D. 5
E. 6

38. A piece of cable x feet in length is cut into exactly 4 pieces, each 3 feet 5 inches in length. What is the value of x?

A. $12\frac{2}{3}$
B. 13
C. $13\frac{1}{3}$
D. $13\frac{2}{3}$
E. 14

39. Which of the following is the product of two positive integers whose sum is 5? Indicate **all** such possible products.

A. 0
B. 2
C. 4
D. 5
E. 6
F. 8
G. 10

40. A 4-pound mixture requires $2\frac{1}{2}$ liters of water. At this rate, how many liters of water should be used for a 7-pound mixture?

LEVEL 2: ALGEBRA

41. Quantity A: $(3x + 1)(3x - 1)$
    Quantity B: $9x^2$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

$$ab = 0$$
$$bc = 2$$

42. Quantity A: a
    Quantity B: 0

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

43. Quantity A: $\frac{c}{3}$
    Quantity B: $\frac{a+b+c}{3}$

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

$$s < 1 \text{ and } t > 15$$

44. Quantity A: $t - s$
    Quantity B: 14

   A. Quantity A is greater.
   B. Quantity B is greater.
   C. The two quantities are equal.
   D. The relationship cannot be determined from the information given.

45. The number a is 5 less than 7 times the number b. The sum of a and b is 14. Which of the following pairs of equations could be used to find the values of a and b?

A. $a = 7(b - 5)$
$a + b = 14$
B. $a = 5(7 - b)$
$a = 14 - b$
C. $a = 7(b - 5)$
$a = 14 - b$
D. $a = 7b - 5$
$a + b = 14$
E. $a = 5 - 7b$
$a + b = 14$

46. If $a^2 + 2ab + b^2 = 7$, then $(a + b)^2 =$

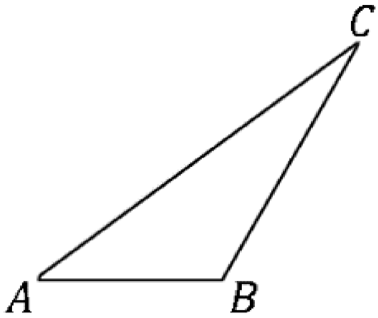
A. $\sqrt{7}$
B. 7
C. $7\sqrt{7}$
D. 49
E. $49\sqrt{7}$

47. Which of the following are solutions to $2x^3 + 2 = 5x - x^2$. Indicate <u>all</u> such solutions.

A. -2
B. -1
C. $-\frac{1}{2}$
D. 0
E. $\frac{1}{2}$
F. 1
G. 2

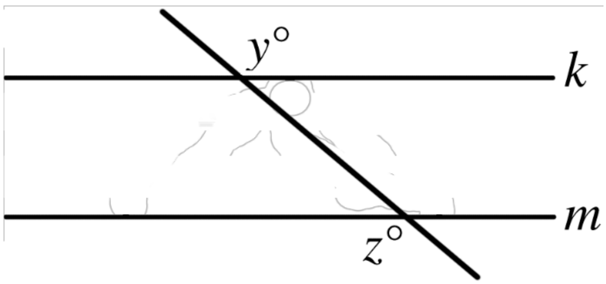
48. If $7x + y = 6$ and $5x + y = 2$, what is the value of $6x + y$?

LEVEL 2: GEOMETRY



49. Quantity A: $AB + BC$
    Quantity B: AC

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



$k \parallel m$

50. Quantity A: y
    Quantity B: z

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

The area of circle O is 64π.

51. Quantity A: The diameter of circle O

    Quantity B: 8

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

The volume of a right circular cylinder is 40π.

52. Quantity A: The height of the cylinder

    Quantity B: The radius of a base of the cylinder

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

53. In the xy-plane, what is the slope of the line whose equation is $2x - 5y = 11$

    A. $-\frac{5}{2}$
    B. $-\frac{11}{5}$
    C. $\frac{2}{5}$
    D. $\frac{5}{2}$
    E. $\frac{11}{2}$

54. C is the midpoint of line segment AB, and D and E are the midpoints of AC and CB, respectively. If the length of DE is 7, what is the length of AB?

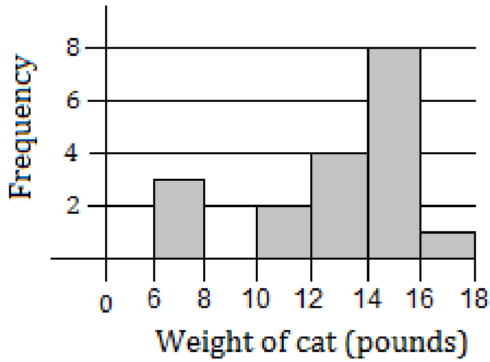
    A. 3.5
    B. 7
    C. 10.5
    D. 14
    E. 17.5

55. In an xy-coordinate system, which points do not lie in the interior of a circle with center $(0,0)$ and radius 5 ? Indicate all such points.

A. $(1,-5)$
B. $(0,5)$
C. $(2,-2)$
D. $(3,3)$
E. $(-3,4)$
F. $(5,5)$

56. The line in the xy-plane that contains the point $(-3,7)$ and $(5,y)$ has slope 1. What is the value of y?

LEVEL 2: DATA ANALYSIS



The histogram above shows the distribution of the weights, in pounds, of 18 cats in a shelter.

57. Quantity A: The median weight of the 18 cats represented in the histogram

Quantity B: 14 pounds

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

A is a set of numbers whose average (arithmetic mean) is 15. B is a set that is generated by multiplying each number in A by six.

58. Quantity A: The arithmetic mean of the numbers in set B

    Quantity B: 90

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

English Exam			Spanish Exam	
Grade	Frequency		Grade	Frequency
100	0		100	1
95	0		95	1
90	6		90	1
85	0		85	1
80	0		80	1
75	0		75	1

59. Quantity A: The standard deviation of grades on the English exam.

    Quantity B: The standard deviation of grades on the Spanish exam.

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

$$16, 18, 4, 20, 3, 6, 23, 37, k$$

k is the median of the 9 numbers listed above.

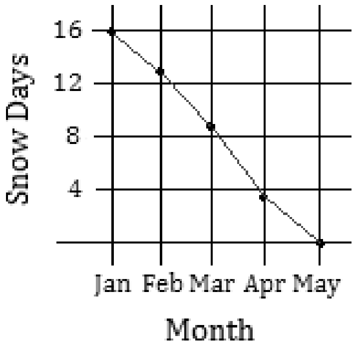
60. Quantity A: k

    Quantity B: 17

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

61. What is the average (arithmetic mean) of $11 - k$, 11, and $11 + k$?

A. 3
B. 11
C. 15
D. $3 + \frac{k}{3}$
E. $11 + \frac{k}{3}$



62. The line graph above shows the average number of days that it snows at least 0.1 inch in Buffalo, NY from January to May. According to the graph, approximately what was the greatest decrease in the number of snow days from one month to the next month?

A. 2
B. 3
C. 4
D. 5
E. 6

$$15, 17, 3, 19, 2, 5, 22, 36, b$$

63. If $30 < b < 40$, then which of the following could be the range of the list above? Indicate <u>all</u> such values.

A. 28
B. 30
C. 32
D. 34
E. 36
F. 38

64. Of the marbles in a jar, 14 are green. Joseph randomly takes one marble out of the jar. If the probability is $\frac{7}{8}$ that the marble he chooses is green, how many marbles are in the jar?

LEVEL 3: ARITHMETIC

65. Quantity A: $\dfrac{1}{3+\frac{1}{2+\frac{1}{2}}}$

Quantity B: $\dfrac{1}{2+\frac{1}{3+\frac{1}{2}}}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$k = 3021.615$$

66. Quantity A: $\dfrac{k\times10^4}{10^7}$

Quantity B: 3

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

67. Quantity A: $\dfrac{\sqrt{135}}{\sqrt{15}}$

Quantity B: $\dfrac{\sqrt{117}}{\sqrt{13}}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

68. Quantity A: 32 increased by 72%
    Quantity B: 72 decreased by 32%

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

69. $\frac{2}{\frac{2}{5}} + \frac{3}{\frac{3}{7}} + \frac{1}{\frac{1}{2}} =$

    A. $\frac{1}{14}$
    B. $\frac{134}{385}$
    C. $\frac{385}{134}$
    D. 134
    E. 14

70. If an integer n is divisible by 10, and 25, what is the next larger integer divisible by these numbers?

    A. $n + 5$
    B. $n + 10$
    C. $n + 25$
    D. $n + 35$
    E. $n + 50$

71. A bag is filled with black and red marbles so that the ratio of the number of black marbles to the number of red marbles is 3 to 7. Which of the following could be the number of marbles in the bag? Indicate all such values.

    A. 20
    B. 23
    C. 25
    D. 27
    E. 30
    F. 33
    G. 35
    H. 37

72. If p is the greatest prime factor of 46 and q is the smallest prime factor of 65, what is the value of $p - q$?

LEVEL 3: ALGEBRA

$$x \neq 0, y > 0$$

73. Quantity A: $\frac{xy}{|xy|}$

Quantity B: 1

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$k > 0, \frac{k}{a} = 321 \text{ and } \frac{k}{b} = 322$$

74. Quantity A: a

Quantity B: b

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

75. Quantity A: 5^x

Quantity B: 6^x

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$z^2 + 2z - 15 = 0$$

76. Quantity A: z^2

Quantity B: 20

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

77. If $5^x = 26$, then $5^{2x} =$

A. 5.2
B. 52
C. 676
D. 1881
E. 11,881,376

78. If $a \neq 0$, then $\frac{a(a^5)^3}{a^7} =$

A. a^7
B. a^8
C. a^9
D. a^{10}
E. a^{11}

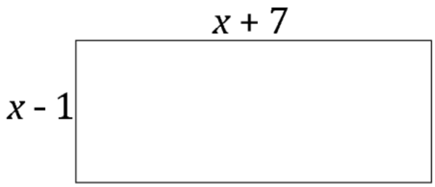
$$\frac{5}{\sqrt{x-7}} = 6$$

79. For $x > 7$, which of the following equations is equivalent to the equation above? Indicate <u>all</u> such equations.

A. $25 = 36(x - 7)$
B. $25 = 36x - 7$
C. $25 = 36x - 252$
D. $25 = 6(x - 7)$
E. $25 = 6x - 7$
F. $25 = 6x - 42$
G. $25 = 6(x - \sqrt{7})$
H. $5 = 6(x - 7)$
I. $5 = 6x - 7$

80. Let h be a function such that $h(x) = |5x| + c$ where c is a constant. If $h(3) = -4$, what is the value of $h(-6)$?

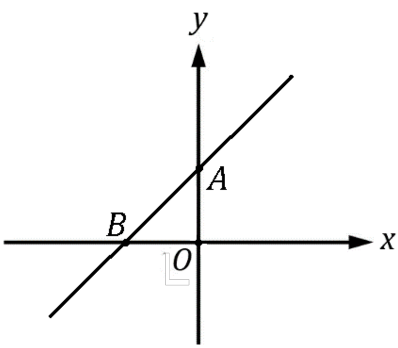
LEVEL 3: GEOMETRY



The perimeter of equilateral triangle T is equal to the perimeter of the rectangle above.

81. Quantity A: The length of a side of T
    Quantity B: $x + 4$

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



The equation of the line graphed in the xy-plane above is $y = \frac{15}{16}x + 4$

82. Quantity A: AO
    Quantity B: BO

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

In the xy-plane, line n passes through the points $(0,-3)$ and $(-1,0)$.

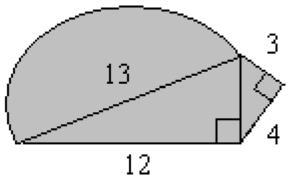
83. Quantity A: The slope of line n

    Quantity B: The slope of a line perpendicular to line n

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

84. Quantity A: The area of a square whose diagonal has length $5\sqrt{2}$?
    Quantity B: 25

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

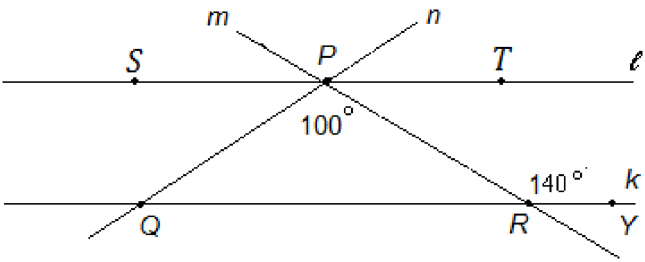


85. What is the total area of the shaded region above to the nearest integer?

    A. 98
    B. 99
    C. 100
    D. 102
    E. 103

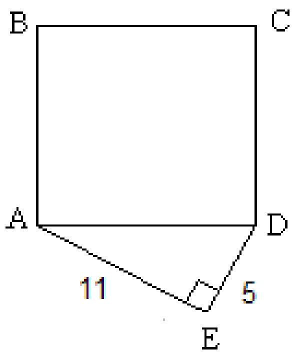
86. Which of the following is an equation of the line in the xy-plane that passes through the point $(0,-3)$ and is perpendicular to the line $y=-4x+7$?

    A. $y=-4x-6$
    B. $y=-4x-3$
    C. $y=-4x+3$
    D. $y=\frac{1}{4}x-3$
    E. $y=\frac{1}{4}x+6$



87. In the figure above, line ℓ is parallel to line k. Transversals m and n intersect at point P on ℓ and intersect k at points R and Q, respectively. Point Y is on k, points S and T are on ℓ, the measure of $\angle PRY$ is 140°, and the measure of $\angle QPR$ is 100°. Which of the following angles have measure 40°? Indicate <u>all</u> such angles.

A. $\angle PRQ$
B. $\angle PQR$
C. $\angle QPS$
D. $\angle RPT$



88. In the figure above, what is the area of square $ABCD$?

Level 3: Data Analysis

89. Quantity A: The number of ways to choose 2 colors from 5
Quantity B: The number of ways to choose 3 colors from 5

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

The average (arithmetic mean) of eleven numbers is 15. When a twelfth number is added, the average of the twelve numbers is 14.

90. Quantity A: The twelfth number

    Quantity B: 4

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

TEST GRADES OF STUDENTS IN MATH CLASS

Test Grade	65	81	84	92	100
Number of students with that grade	1	8	6	2	4

The test grades of the 21 students in a math class are shown in the chart above.

91. Quantity A: The median of the test grades

    Quantity B: The mode of the test grades

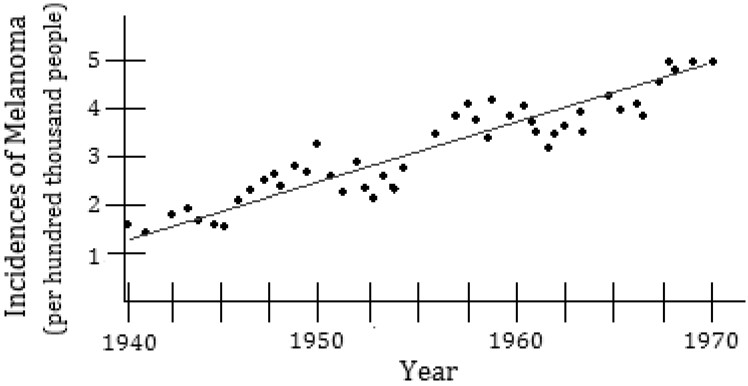
    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

Seven different books are to be stacked in a pile. One book is chosen for the bottom of the pile.

92. Quantity A: The number different orders the remaining books can be placed on the stack?

    Quantity B: 17

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.



93. The scatterplot above shows the numbers of incidences of melanoma, per 100,000 people from 1940 to 1970. Based on the line of best fit to the data, as shown in the figure, which of the following values is closest to the average yearly increase in the number of incidences of melanoma?

A. 33,000
B. 23,000
C. 13,000
D. 0.33
E. 0.13

94. There are y bricks in a row. If one brick is to be selected at random, the probability that it will be cracked is $\frac{3}{11}$. In terms of y, how many of the bricks are <u>not</u> cracked?

A. $\frac{y}{11}$
B. $\frac{8y}{11}$
C. $\frac{11y}{8}$
D. $\frac{3y}{11}$
E. $11y$

95. The set Q consists of 15 numbers whose arithmetic mean is zero? Which of the following quantities must also be zero? Indicate <u>all</u> such quantities.

A. The median of the numbers in Q
B. The mode of the numbers in Q
C. The sum of the numbers in Q

96. A chemist is testing 6 different liquids. For each test, the chemist chooses 2 of the liquids and mixes them together. What is the least number of tests that must be done so that every possible combination of liquids is tested?

LEVEL 4: ARITHMETIC

97. Quantity A: $\frac{35^7}{5^{77}}$

Quantity B: $\frac{7^7}{5^{70}}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

If k is divided by 8, the remainder is 3.

98. Quantity A: The remainder when $3k$ is divided by 8
Quantity B: 2

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

a and b are positive integers, and $110a + 17b = 4417$

99. Quantity A: $a + b$
Quantity B: 41

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

The product of five consecutive integers written in increasing order equals the second integer.

100. Quantity A: The greatest of the five integers
Quantity B: 2

    A. Quantity A is greater.
    B. Quantity B is greater.
    C. The two quantities are equal.
    D. The relationship cannot be determined from the information given.

101. The number $.0001$ is how many times as great as the number $(0.000001)^2$?

    A. 10^4
    B. 10^6
    C. 10^8
    D. 10^{10}
    E. 10^{12}

102. n is a two-digit number whose units digit is 3 times its tens digit, which of the following statements must be true?

    A. n is less than 15
    B. n is greater than 30
    C. n is a multiple of 3
    D. n is a multiple of 10
    E. n is a multiple of 13

103. If m and n are distinct positive integers such that n is divisible by m, and m is divisible by 3, which of the following statements must be true? Indicate <u>all</u> such statements.

    A. n is divisible by 3
    B. $n = 3m$
    C. n has more than 3 positive factors

104. A certain exam lasts a total of 4 hours. Each part of the exam requires the same amount of time, and 10 minute breaks are included between consecutive parts. if there are a total of 4 breaks during the 4 hours, what is the required time, in minutes, for each part of the test?

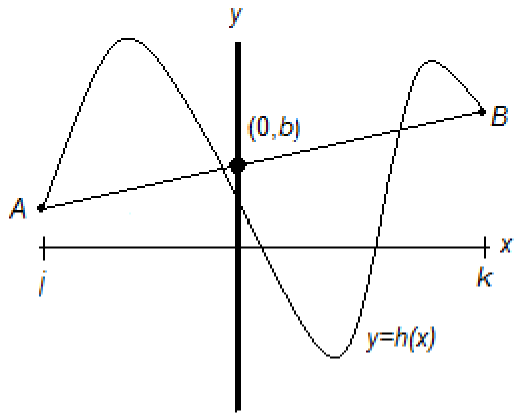
LEVEL 4: ALGEBRA

$$a < b$$

105. Quantity A: $\frac{a}{5}$

Quantity B: $\frac{b}{4}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.



The figure above shows the graph of the function h and line segment $\overline{AB}$, which has a y-intercept of $(0, b)$.

106. Quantity A: The number of values of x between j and k for which $h(x) = b$

Quantity B: 4

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$15 < |b - 11| < 16 \text{ and } b < 0$$

107. Quantity A: $|b|$
Quantity B 4.5

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$x^2 = 9 \text{ and } y^2 = 5$$

108. Quantity A: $(2x + y)^2$
Quantity B: 65

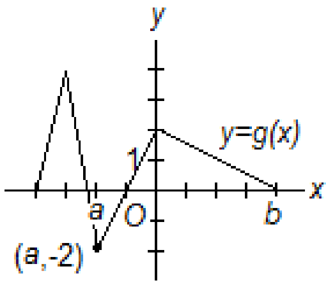
A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

109. The function g has the property that $g(a) = g(b)$ for all real numbers a and b. What is the graph of $y = g(x)$ in the xy-plane?

A. A parabola symmetric about the x-axis
B. A line with slope 0
C. A line with slope 1
D. A line with no slope
E. A semicircle centered at the origin

110. If $a \neq 13$ and $\frac{a^2-169}{a-13} = b^2$, what does a equal in terms of b ?

A. $b^2 - 13$
B. $b^2 + 13$
C. $\sqrt{b} - \sqrt{13}$
D. $b - \sqrt{13}$
E. $b + \sqrt{13}$

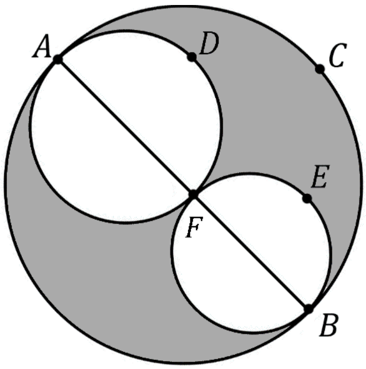


111. The figure above shows the graph of the function g in the xy-plane. Which of the following statements are true? Indicate all such statements.

A. $g(b) = 0$
B. $g(a) + g(b) + g(0) = 0$
C. $g(a) > g(b)$

112. If $|x - 7| = 19$ and 6 is a divisor of x, what is the value of x ?

LEVEL 4: GEOMETRY



Three circles with their centers on line segment AB are tangent at points A, F, and B, where point F lies on line segment AB.

113. Quantity A: The area of the shaded region
Quantity B: The area of the unshaded region

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

In the triangle RST, $RS = RT = 26$ and $ST = 20$

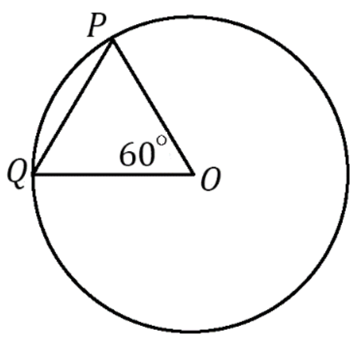
114. Quantity A: The area of triangle RST
Quantity B: 250

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

A square has a side of length $x + 5$ and a diagonal of length $x + 10$.

115. Quantity A: x
Quantity B: 5

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.



O is the center of the circle and the area of ΔPOQ is $4\sqrt{3}$.

116. Quantity A: The area of the circle
Quantity B: The circumference of the circle

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

117. If $a > 1$, what is the slope of the line in the xy-plane that passes through the points (a^2, a^4) and (a^3, a^6)?

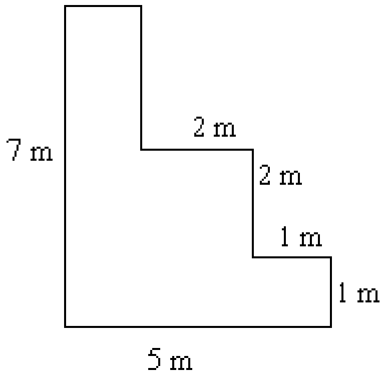
A. $-a^3 + 6a^2$
B. $-a^3 + a^2$
C. $-a^3 - a^2$
D. $a^3 - a^2$
E. $a^3 + a^2$

118. A container in the shape of a right circular cylinder has an inside base diameter of 10 centimeters and an inside height of 6 centimeters. This cylinder is completely filled with fluid. All of the fluid is then poured into a second right circular cylinder with a larger inside base diameter of 14 centimeters. What must be the minimum inside height, in centimeters, of the second container?

A. $\frac{5}{\sqrt{7}}$
B. $\frac{7}{5}$
C. 5
D. $\frac{150}{49}$
E. $2\sqrt{7}$

119. Points P and Q are on the surface of a sphere that has a volume of 972π cubic meters. Which of the following are possible lengths, in meters, of line segment $\overline{PQ}$? (The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.) Indicate **all** such values.

A. 4
B. 12
C. 16
D. 17
E. 18
F. 19
G. 20



120. Let P be the perimeter of the figure above in meters, and let A be the area of the figure above in square meters. What is the value of $P + A$?

LEVEL 4: DATA ANALYSIS

Exactly 4 musicians try out to play 4 different instruments for a particular performance. Each musician can play each of the 4 instruments, and each musician is assigned an instrument.

121. Quantity A: The probability that Jerry will play the tuba
Quantity B: 0.25

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

A farmer purchased several animals from a neighboring farmer: 6 animals costing \$50 each, 10 animals costing \$100 each, and k animals costing \$200 each, where k is a positive odd integer. The median price for all the animals was \$100.

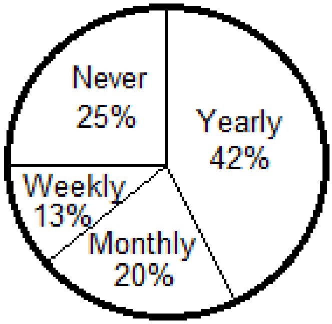
122. Quantity A: k
Quantity B: 16

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

The average (arithmetic mean) of a, $2a$, b, and $4b$ is $2a$.

123. Quantity A: a
Quantity B: b

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.



The circle graph above shows the distribution of responses to a survey in which a group of people were asked how often they donate to charity.

124. Quantity A: The fraction of those surveyed that reported they donate at least yearly
Quantity B: $\frac{21}{50}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

125. The average (arithmetic mean) of 17 numbers is j. If two of the numbers are k and m, what is the average of the remaining 15 numbers in terms of j, k and m?

A. $\frac{k+m}{17}$
B. $17j + k + m$
C. $\frac{16j-k-m}{17}$
D. $\frac{17j-k-m}{15}$
E. $\frac{17(k-m)-j}{15}$

126. If the average (arithmetic mean) of k and $k + 5$ is b and if the average of k and $k - 9$ is c, what is the average of b and c?

A. $k - 2$
B. $k - 1$
C. k
D. $k + \frac{1}{2}$
E. $2k$

127. On a certain exam, the median grade for a group of 25 students is 67. If the highest grade on the exam is 90, which of the following could be the number of students that scored 67 on the exam? Indicate <u>all</u> such values.

A. 5
B. 20
C. 24

128. The integers 1 through 5 are written on each of five cards. The cards are shuffled and one card is drawn at random. That card is then replaced, the cards are shuffled again and another card is drawn at random. This procedure is repeated one more time (for a total of three times). What is the probability that the sum of the numbers on the three cards drawn was between 13 and 15, inclusive? Express your answer as a fraction.

LEVEL 5: ARITHMETIC

$$0 > c > d$$

129. Quantity A: cd
Quantity B: $(cd)^3$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$$m > 0, n > 0, k > 0$$

130. Quantity A: $m + \frac{1}{n+\frac{1}{k}}$
Quantity B: $\frac{mnk+m+k}{nk+1}$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

131. $\left(\sqrt{2} - \sqrt{3}\right)^2 =$

A. 1
B. $5 - 2\sqrt{3}$
C. $5 - 2\sqrt{6}$
D. $1 - \sqrt{6}$
E. $1 - 2\sqrt{6}$

132. How many positive integers are both multiples of 3 and divisors of 243?

A. Two
B. Three
C. Four
D. Five
E. Six

133. If $n \leq -5$. which of the following has the least value?

A. $\frac{1}{(n+3)^2}$
B. $-\frac{1}{(n+3)^2}$
C. $\frac{1}{n+3}$
D. $\frac{1}{n+4}$
E. $\frac{1}{n-4}$

134. If $a^8b^7c^6d^5 > 0$, which of the following products must be positive? Indicate <u>all</u> such products.

A. ab
B. ac
C. ad
D. bc
E. bd
F. cd

135. If $9 \leq x \leq 15$ and $3 \leq y \leq 5$, what is the greatest possible value of $\frac{7}{x-y}$? Express your answer as a fraction.

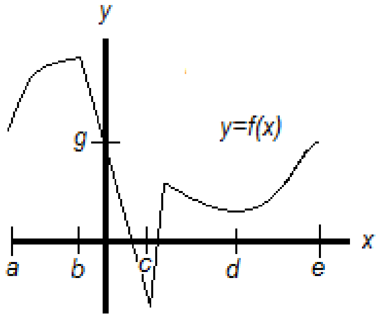
136. The integer k is equal to m^2 for some integer m. If k is divisible by 20 and 24, what is the smallest possible positive value of k?

LEVEL 5: ALGEBRA

$$xy = 22, yz = 10, xz = 55, x > 0$$

137. Quantity A: xyz
Quantity B: 111

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.



138. Quantity A: $f(b) - f(c)$
Quantity B: $f(a) - f(d)$

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

139. If $x^2 + y^2 = k^2$, and $xy = 8 - 4k$, what is $(x + y)^2$ in terms of k?

A. $k - 4$
B. $(k - 4)^2$
C. $k^2 - 4k + 8$
D. $(k - 2)^2 + 4$
E. $(k + 4)^2$

140. If a and b are positive integers, which of the following is equivalent to $(7a)^{5b} - (7a)^{2b}$?

A. $(7a)^{3b}$
B. $7^b(a^5 - a^2)$
C. $(7a)^{2b}[(7a)^{3b} - 1]$
D. $(7a)^{2b}[49a^b - 1]$
E. $(7a)^{2b}[(7a)^5 - 1]$

141. If $f(x) = x^2 - 5$, which of the following statements are true? Indicate all such statements.

A. $f(-3) = |f(-3)|$
B. $f(-2) = -|f(-2)|$
C. $f(1) < |f(-1)|$
D. $f(0) = |f(0)|$
E. $f(2) < |f(2)|$

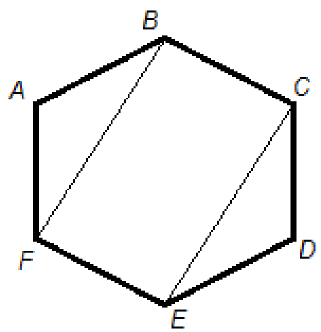
142. If $\frac{x+2y}{2x-y} = 3$ and $y = -1$, what is the value of x ?

$$k = a - b + 12$$
$$k = b - c - 17$$
$$k = c - a + 11$$

143. In the system of equations above, what is the value of k ?

144. Let $x \therefore y$ be defined as the sum of all integers between x and y. For example, $1 \therefore 4 = 2 + 3 = 5$. What is the value of $(60 \therefore 900) - (63 \therefore 898)$?

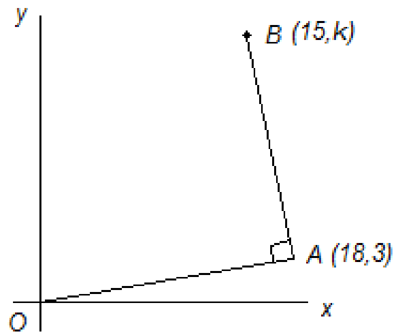
LEVEL 5: GEOMETRY



$ABCDEF$ is a regular hexagon and $CD = 6$.

145. Quantity A: The perimeter of rectangle $BCEF$
Quantity B: 33

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.



$OA = AB$

146. Quantity A: k
Quantity B 18

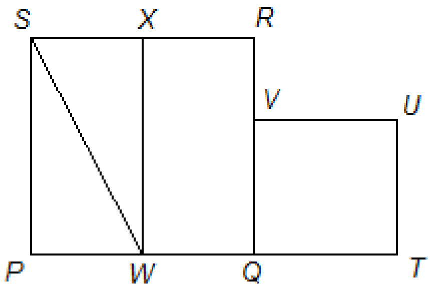
A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

147. How many solid wood cubes, each with a total surface area of 294 square centimeters, can be cut from a solid wood cube with a total surface area of 2,646 square centimeters if no wood is lost in the cutting?

A. 3
B. 9
C. 27
D. 81
E. 243

148. In the xy-plane, the points $(5, e)$ and $(f, 7)$ are on a line that is perpendicular to the graph of the line $y = -\frac{1}{5}x + 12$. Which of the following represents e in terms of f?

A. $5f + 32$
B. $-5f + 32$
C. $5f + 25$
D. $-\frac{1}{5}f + 32$
E. $\frac{1}{5}f + 32$



149. In the figure above, $PQRS$ and $QTUV$ are squares, W and X are the midpoints of $\overline{PQ}$ and $\overline{RS}$, respectively, and $TW = SW$. If $RX = \frac{1}{2}$, what is the length of $\overline{UV}$?

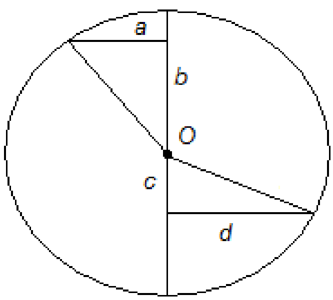
A. $\frac{\sqrt{5}-1}{2}$

B. $\frac{\sqrt{3}-1}{2}$

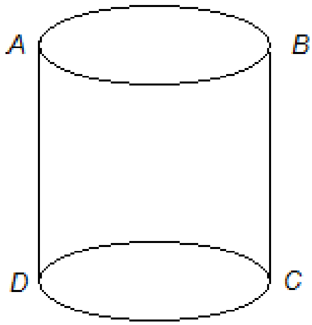
C. $\frac{\sqrt{5}}{2}$

D. $\frac{\sqrt{3}}{2}$

E. $\frac{2}{3}$



150. In the figure above, O is the center of the circle, the two triangles have legs of lengths a, b, c, and d, $a^2 + b^2 + c^2 + d^2 = 15$, and the area of the circle is $k\pi$. What is the value of k ? Express your answer as a fraction.



151. The figure shown above is a right circular cylinder. The circumference of each circular base is 20, the length of AD is 14, and AB and CD are diameters of each base respectively. If the cylinder is cut along AD, opened, and flattened, what is the length of AC to the nearest tenth?

152. A sphere with volume 36π cubic inches is inscribed in a cube so that the sphere touches the cube at 6 points. What is the surface area, in square inches, of the cube? (The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

LEVEL 5: DATA ANALYSIS

A group of students take a test and the average score is 90. One more student takes the test and receives a score of 81 decreasing the average score of the group to 87.

153. Quantity A: The number of students in the initial group
Quantity B: 3

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

$a < b < c$, the average of a and b is 7, the average of b and c is 11, the average of a and c is 10. What is the average of a, b and c?

154. Quantity A: The average of a, b and c
Quantity B: 9.3

A. Quantity A is greater.
B. Quantity B is greater.
C. The two quantities are equal.
D. The relationship cannot be determined from the information given.

155. One hundred cards numbered 200 through 299 are placed into a bag. After shaking the bag, 1 card is randomly selected from the bag. Without replacing the first card, a second card is drawn. If the first card drawn is 265, what is the probability that both cards drawn have the same tens digit?

A. $\frac{1}{8}$
B. $\frac{1}{9}$
C. $\frac{1}{10}$
D. $\frac{1}{11}$
E. $\frac{1}{99}$

156. Suppose that the average (arithmetic mean) of a, b, and c is h, the average of b, c, and d is j, and the average of d and e is k. What is the average of a and e?

A. $h - j + k$
B. $\frac{3h+3j-2k}{2}$
C. $\frac{3h-3j+2k}{2}$
D. $\frac{3h-3j+2k}{5}$
E. $\frac{3h-3j+2k}{8}$

157. A set of marbles contains only black marbles, white marbles, and yellow marbles. If the probability of randomly choosing a black marble is $\frac{1}{14}$ and the probability of randomly choosing a white marble is $\frac{3}{4}$, what is the probability of randomly choosing a yellow marble?

A. $\frac{5}{28}$

B. $\frac{3}{14}$

C. $\frac{1}{4}$

D. $\frac{2}{7}$

E. $\frac{9}{28}$

158. How many integers between 4000 and 7000 have digits that are all different and that increase from left to right?

159. Seven cards, each of a different color are shuffled and placed in a row. What is the probability that the blue card is placed at an end? Express your answer as a fraction.

160. The average (arithmetic mean) salary of employees at an advertising firm with P employees in thousands of dollars is 53, and the average salary of employees at an advertising firm with Q employees in thousands of dollars is 95. When the salaries of both firms are combined, the average salary in thousands of dollars is 83. What is the value of $\frac{P}{Q}$? Express your answer as a fraction.

Answers to Supplemental Problems

Level 1: Arithmetic

1. C
2. A
3. A
4. B
5. C
6. D
7. B, C
8. 9.2

Level 1: Algebra

9. B
10. A
11. C
12. B
13. A
14. C
15. D
16. 4/3

Level 1: Geometry

17. A
18. C
19. B
20. B
21. D
22. B
23. C, E
24. 28

Level 1: Data Analysis

25. C
26. A
27. A
28. D
29. D
30. C
31. A, D
32. 16

Level 2: Arithmetic

33. D
34. B
35. D
36. A
37. E
38. D
39. C, E
40. 4.375

Level 2: Algebra

41. B
42. C
43. D
44. A
45. D
46. B
47. A, E, F
48. 4

Level 2: Geometry

49. A
50. C
51. A
52. D

53. C
54. D
55. A, B, E, F
56. 15

Level 2: Data Analysis

57. D
58. C
59. B
60. D
61. B
62. E
63. D, E
64. 16

Level 3: Arithmetic

65. B
66. A
67. C
68. A
69. E
70. E
71. A, E
72. 18

Level 3: Algebra

73. D
74. A
75. D
76. D
77. C
78. C
79. A, C
80. 11

Level 3: Geometry

81. A
82. B
83. B
84. C
85. D
86. D
87. A, B, C, D
88. 146

Level 3: Data Analysis

89. C
90. B
91. A
92. A
93. C
94. B
95. C
96. 15

Level 4: Arithmetic

97. C
98. B
99. D
100. A
101. C
102. E
103. A
104. 40

Level 4: Algebra

105. D
106. B
107. D
108. D
109. B

110. A
111. A, B
112. -12

Level 4: Geometry

113. D
114. B
115. A
116. A
117. E
118. D
119. A, B, C, D, E
120. 45

Level 4: Data Analysis

121. C
122. B
123. C
124. A
125. D
126. B
127. A, B, C
128. $2/25$

Level 5: Arithmetic

129. D
130. C
131. C
132. D
133. D
134. E
135. $7/4$
136. 3600

Level 5: Algebra

137. B
138. A
139. B
140. C
141. A, B, C, E
142. -1
143. 2
144. 1983

Level 5: Geometry

145. B
146. A
147. C
148. B
149. A
150. $15/2$
151. 17.2
152. 216

Level 5: Data Analysis

153. B
154. A
155. D
156. C
157. A
158. 15
159. $2/7$
160. $2/5$

About the Author

Dr. Steve Warner, a New York native, earned his Ph.D. at Rutgers University in Pure Mathematics in May, 2001. While a graduate student, Dr. Warner won the TA Teaching Excellence Award.

After Rutgers, Dr. Warner joined the Penn State Mathematics Department as an Assistant Professor. In September, 2002, Dr. Warner returned to New York to accept an Assistant Professor position at Hofstra University. By September 2007, Dr. Warner had received tenure and was promoted to Associate Professor. He has taught undergraduate and graduate courses in Precalculus, Calculus, Linear Algebra, Differential Equations, Mathematical Logic, Set Theory and Abstract Algebra.

Over that time, Dr. Warner participated in a five year NSF grant, "The MSTP Project," to study and improve mathematics and science curriculum in poorly performing junior high schools. He also published several articles in scholarly journals, specifically on Mathematical Logic.

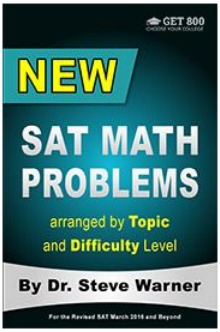
Dr. Warner has more than 15 years of experience in general math tutoring and tutoring for standardized tests such as the SAT, ACT and AP Calculus exams. He has tutored students both individually and in group settings.

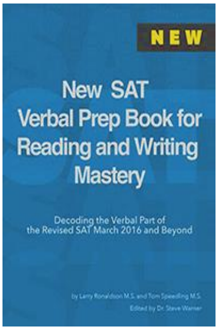
In February, 2010 Dr. Warner released his first SAT prep book "The 32 Most Effective SAT Math Strategies," and in 2012 founded Get 800 Test Prep. Since then Dr. Warner has written books for the SAT, ACT, SAT Math Subject Tests and AP Calculus exams.

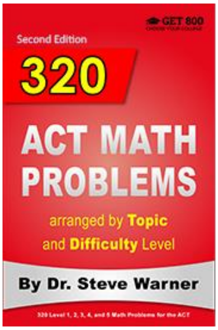
Dr. Steve Warner can be reached at

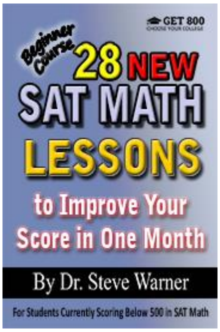
steve@SATPrepGet800.com

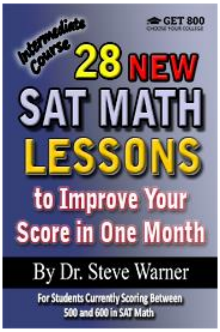
BOOKS BY DR. STEVE WARNER

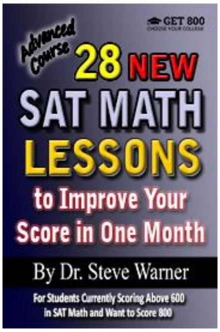


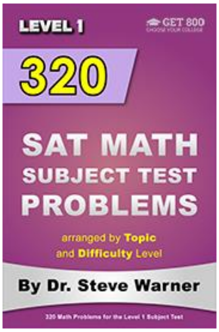


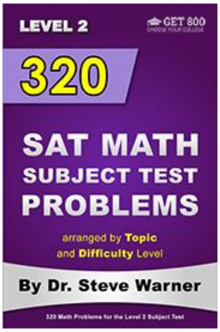


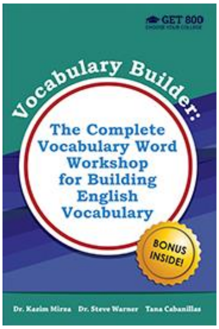












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